Tanaka's equation on the circle and stochastic flows

February 27, 2013

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Abstract

We define a Tanaka's equation on an oriented graph with two edges and two vertices. This graph will be embedded in the unit circle. Extending this equation to flows of kernels, we show that the laws of the flows of kernels K solution of Tanaka's equation can be classified by pairs of probability measures (m^+, m^-) on $[0,1]$, with mean $1/2$. What happens at the first vertex is governed by m^+, and at the second by m^-. For each vertex P, we construct a sequence of stopping times along which the image of the whole circle by K is reduced to P. We also prove that the supports of these flows contains a finite number of points, and that except for some particular cases this number of points can be arbitrarily large.

1 Introduction and main results

Consider Tanaka's equation

$$\varphi_{s,t}(x) = x + \int_s^t \operatorname{sgn}(\varphi_{s,u}(x))dW_u, \quad s \le t, x \in \mathbb{R}, \tag{1}$$

where $(W_t)_{t\in\mathbb{R}}$ is a Brownian motion on $\mathbb{R}$ (that is $(W_t)_{t\ge0}$ and $(W_{-t})_{t\ge0}$ are two independent standard Brownian motions) and φ is a stochastic flow of mappings (see [6] for a precise definition). In [7], Le Jan and Raimond have extended (1) to kernels: if K is a stochastic flow of kernels (see [6]) and

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W is a Brownian motion on $\mathbb{R}$, then (K, W) is said to solve Tanaka's equation if and only if for all $s \le t, x \in \mathbb{R}$, $f \in C_b^2(\mathbb{R})$ (f is C^2 on $\mathbb{R}$ and f', f'' are bounded), a.s.

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(f'\mathrm{sgn})(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du. \tag{2}$$

The main result of [7] is a one-to-one correspondence between probability measures m on $[0, 1]$ with mean $\frac{1}{2}$ and laws of solutions to (2). Denote by $\mathbb{P}^m$, the law of the solution (K, W) associated to m. Then

$$K_{s,t}(x) = \delta_{x+\mathrm{sgn}(x)W_{s,t}} 1_{\{t\le\tau_{s,x}\}} + (U_{s,t}\delta_{W^+_{s,t}} + (1 - U_{s,t})\delta_{-W^+_{s,t}})1_{\{t>\tau_{s,x}\}}$$

where $W_{s,t} = W_t - W_s,\ W^+_{s,t} = W_t - \inf_{u\in[s,t]} W_u = W_{s,t} - \inf_{u\in[s,t]} W_{s,u}$,

$$\tau_{s,x} = \inf\{t \ge s : W_{s,t} = -|x|\}$$

and where $U_{s,t}$ is independent of W, with law m. In particular, when $m = \delta_{\frac{1}{2}}$, then $U_{s,t} = \frac{1}{2}$ and K is $\sigma(W)$-measurable; this is also the unique $\sigma(W)$-measurable solution of (2). For $m = \frac{1}{2}(\delta_0 + \delta_1)$, we recover the unique flow of mappings solving (1) which was firstly introduced in [9]. In [2], a more general Tanaka's equation has been defined on a graph related to Walsh's Brownian motion. In this work, we deal with another simple oriented graph with two edges and two vertices that will be embedded in the unit circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$.

A function f defined on $\mathscr{C}$ is said to be derivable in $z_0 \in \mathscr{C}$ if

$$f'(z_0) := \lim_{h\to 0} \frac{f(z_0e^{ih}) - f(z_0)}{h}$$

exists. Let $C^2(\mathscr{C})$ be the space of all functions f defined on $\mathscr{C}$ having first and second continuous derivatives f' and f''. Let $\mathcal{P}(\mathscr{C})$ be the space of all probability measures on $\mathscr{C}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions dense in $\{f \in C(\mathscr{C}), ||f||_\infty \le 1\}$. We equip $\mathcal{P}(\mathscr{C})$ with the following distance d and its associated Borel σ-field:

$$d(\mu, \nu) = \left(\sum_n 2^{-n}\left(\int f_n d\mu - \int f_n d\nu\right)^2\right)^{\frac{1}{2}} \text{ with } \mu, \nu \in \mathcal{P}(\mathscr{C}). \tag{3}$$

In the following, $\arg(z) \in [0, 2\pi[$ denotes the argument of $z \in \mathbb{C}$ and in all the paper l is a fixed parameter in $]0, \pi]$. Define for $z \in \mathscr{C}$,

$$\epsilon(z) = 1_{\{arg(z)\in[0,l]\}} - 1_{\{arg(z)\in]l,2\pi[\}}$$

and denote by $\mathscr{C}_l$ (or simply by $\mathscr{C}$ since l will not vary) the graph embedded in $\mathscr{C}$ with two vertices 1 and e^{il} and two edges $\mathscr{C}^+ = \{z \in \mathscr{C} : arg(z) \in]0, l[\}$ and $\mathscr{C}^- = \mathscr{C} \setminus \mathscr{C}^+$ with orientation given by ε (see Figure 1 below).

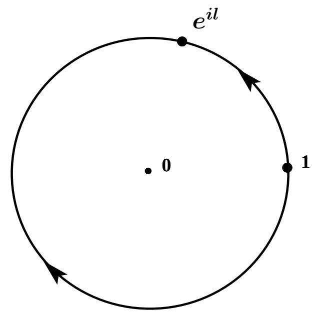

Figure 1: The graph $\mathscr{C}$.

Definition 1. *On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let W be a Brownian motion on $\mathbb{R}$ and K be a stochastic flow of kernels on $\mathscr{C}$. We say that (K, W) solves Tanaka's equation on $\mathscr{C}$ denoted $(T_{\mathscr{C}})$ if for all $s \le t, f \in C^2(\mathscr{C}), x \in \mathscr{C}$, as.*

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\epsilon f')(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du. \tag{4}$$

If (K, W) is a solution of $(T_{\mathscr{C}})$ and $K = \delta_\varphi$ with φ a stochastic flow of mappings, we simply say that (φ, W) solves $(T_{\mathscr{C}})$.

If (K, W) is a solution of $(T_{\mathscr{C}})$, then following Lemma 3.1 of [7], we have $\sigma(W) \subset \sigma(K)$ (see Lemma 3 (ii) below). So we will simply say that K solves $(T_{\mathscr{C}})$.

In this paper, given two probability measures on $[0, 1]$, m^+ and m^- with mean $\frac{1}{2}$, we construct a flow K^{m^+,m^-} solution of $(T_{\mathscr{C}})$. Let (K^+, K^-, W) be such that given W, K^+ and K^- are independent and $(K^\pm, \pm W)$ has for law $\mathbb{P}^{m^\pm}$. The flows K^+ and K^- provide the additional randomness when K^{m^+,m^-} passes through 1 or e^{il}. Away from these two points, K^{m^+,m^-} just follows W on $\mathscr{C}^+$ and $-W$ on $\mathscr{C}^-$. We now state our first result.

Theorem 1. *(1) Let m^+ and m^- be two probability measures on $[0, 1]$ satisfying*

$$\int_0^1 u\ m^+(du) = \int_0^1 u\ m^-(du) = \frac{1}{2}. \tag{5}$$

There exists a stochastic flow of kernels (unique in law) K^{m^+,m^-} and a Brownian motion W on $\mathbb{R}$ such that (K^{m^+,m^-}, W) solves $(T_{\mathscr{C}})$ and such that if $W^+_{s,t} = W_t - \inf_{u\in[s,t]} W_u, W^-_{s,t} = \sup_{u\in[s,t]} W_u - W_t$ and

$$\rho_s = \inf\{t \ge s,\ \sup(W^+_{s,t}, W^-_{s,t}) = l\},$$

then conditionally to $\{s \le t < \rho_s\}$, *a.s.*

$$\begin{aligned} K^{m^+,m^-}_{s,t}(1) &= U^+_{s,t}\delta_{\exp(iW^+_{s,t})} + (1-U^+_{s,t})\delta_{\exp(-iW^+_{s,t})}, \\ K^{m^+,m^-}_{s,t}(e^{il}) &= U^-_{s,t}\delta_{\exp(i(l+W^-_{s,t}))} + (1-U^-_{s,t})\delta_{\exp(i(l-W^-_{s,t}))} \end{aligned}$$

and conditionally to $\{s \le t < \rho_s\}$, $(U^+_{s,t}, U^-_{s,t})$ *is independent of* W *and has for law* $m^+ \otimes m^-$.
(2) For all K *solution of* $(T_{\mathscr{C}})$, *there exists a unique pair of probability measures* (m^+, m^-) *satisfying (5) such that* $K \stackrel{law}{=} K^{m^+,m^-}$.

The version (K^{m^+,m^-}, W) of the solution of $(T_{\mathscr{C}})$ defined in Theorem 1 (1), and constructed in section 2, satisfies Proposition 1 and Proposition 2 below. For all $-\infty \le s \le t \le +\infty$, let

$$\mathcal{F}^W_{s,t} = \sigma(W_{u,v},\ s \le u \le v \le t). \tag{6}$$

Proposition 1. *(1) There exists an increasing sequence* $(S_k)_{k\ge1}$ *of* $(\mathcal{F}^W_{0,t})_{t\ge0}$*-stopping times such that a.s.* $\lim_{k\to\infty} S_k = +\infty$ *and* $K^{m^+,m^-}_{0,S_k}(z) = \delta_{e^{il}}$ *for all* $z \in \mathscr{C}$ *and all* $k \ge 1$.
(2) There exists an increasing sequence $(T_k)_{k\ge1}$ *of* $(\mathcal{F}^W_{0,t})_{t\ge0}$*-stopping times such that a.s.* $\lim_{k\to\infty} T_k = +\infty$ *and* $K^{m^+,m^-}_{0,T_k}(z) = \delta_1$ *for all* $z \in \mathscr{C}$ *and all* $k \ge 1$.

The second proposition shows that the support of K^{m^+,m^-} may contain an arbitrary large number of points with positive probability (more informations can be found in Section 5).

Proposition 2. *Asumme that* m^+ *and* m^- *are both distinct from* $\frac{1}{2}(\delta_0 + \delta_1)$. *Then there exists a sequence of events* $(C_n)_{n\ge0}$ *and a sequence of* $(\mathcal{F}^W_{0,t})_{t\ge0}$*-stopping times* $(\sigma_n)_{n\ge0}$ *such that for all* $n \ge 0$,

- *(i)* $\mathbb{P}(C_n) > 0$,
- *(ii) Card supp* $\left(K^{m^+,m^-}_{0,\sigma_n}(1)\right) = n+1$ *a.s. on* C_n.

We also mention that all the sequences of stopping times discussed in the previous two propositions will be constructed independently of (m^+, m^-). They take values in $\{\rho_n, n \in \mathbb{N}\}$ where $\rho_0 = 0$ and $\rho_{n+1} = \inf\{t \ge \rho_n,\ \sup(W^+_{\rho_n,t}, W^-_{\rho_n,t}) = l\}$ for $n \ge 0$. Set, for $z \in \mathscr{C}$,

$$X_n = \operatorname{supp}\left(K^{m^+,m^-}_{0,\rho_n}(z)\right)$$

where m^+ and m^- are distinct from $\frac{1}{2}(\delta_0 + \delta_1)$. Then $(X_n)_n$ is a strong Markov chain on $E = \cup_{k\ge1}\mathscr{C}^k$. Proposition 1 asserts that $\{1\}$ and $\{e^{il}\}$ are recurrent for this chain. Proposition 2 asserts that for all $n \ge 0$, both $\{1\}$ and $\{e^{il}\}$ communicate with $\mathscr{C}^{n+1}$. So one can deduce the following immediate

Corollary 1. *For all $n \geq 0$, $\mathscr{C}^{n+1}$ is a recurrent set for X (i.e. a.s. $\forall n \geq 0,\ X_k \in \mathscr{C}^{n+1}$ for infinetly many k).*

The paper is organized as follows. In Section 2, we prove the first part of Theorem 1. The proof of the second part will be the subject of Section 3. In Section 4, we prove Proposition 1. Section 5 gives some informations about the support of K^{m^+,m^-} and proves Proposition 2.

2 Construction of flows associated to $(T_{\mathscr{C}})$

Fix two probability measures m^+ and m^- on $[0,1]$ with mean $\frac{1}{2}$.

2.1 Coupling flows associated with two Tanaka's equations on $\mathbb{R}$

In this section, we follow [7]. By Kolmogorov extension theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which one can construct a process $(\varepsilon^+_{s,t}, \varepsilon^-_{s,t}, U^+_{s,t}, U^-_{s,t}, W_{s,t})_{-\infty<s\leq t<\infty}$ taking values in $\{-1,1\}^2 \times [0,1]^2 \times \mathbb{R}$ such that (i), (ii), (iii) and (iv) are satisfied, where

(i) $W_{s,t} := W_t - W_s$ for all $s \leq t$ and W is a Brownian motion on $\mathbb{R}$.

(ii) Given W, $(\varepsilon^+_{s,t}, U^+_{s,t})_{s\leq t}$ and $(\varepsilon^-_{s,t}, U^-_{s,t})_{s\leq t}$ are independent.

(iii) For fixed $s < t$, $(\varepsilon^\pm_{s,t}, U^\pm_{s,t})$ is independent of W and

$$(\varepsilon^\pm_{s,t}, U^\pm_{s,t}) \overset{law}{=} (u\delta_1(dx) + (1-u)\delta_{-1}(dx))m^\pm(du).$$

In particular $\mathbb{P}(\varepsilon^\pm_{s,t} = 1 | U^\pm_{s,t}) = U^\pm_{s,t}$.

(iv) Define for all $s \leq t$

$$\mathrm{m}^+_{s,t} = \inf\{W_u; u \in [s,t]\}, \quad \mathrm{m}^-_{s,t} = \sup\{W_u; u \in [s,t]\}.$$

For all $s < t$ and $\{(s_i, t_i); 1 \leq i \leq n\}$ with $s_i < t_i$, the law of $(\varepsilon^\pm_{s,t}, U^\pm_{s,t})$ knowing $(\varepsilon^\pm_{s_i,t_i}, U^\pm_{s_i,t_i})_{1\leq i\leq n}$ and W is given by

$$(u\delta_1(dx) + (1-u)\delta_{-1}(dx))m^\pm(du)$$

when $\mathrm{m}^\pm_{s,t} \notin \{\mathrm{m}^\pm_{s_i,t_i}; 1 \leq i \leq n\}$ and is given by

$$\sum_{i=1}^n \delta_{\varepsilon^\pm_{s_i,t_i}, U^\pm_{s_i,t_i}} \times \frac{1_{\{\mathrm{m}^\pm_{s,t} = \mathrm{m}^\pm_{s_i,t_i}\}}}{\mathrm{Card}\{i; \mathrm{m}^\pm_{s_i,t_i} = \mathrm{m}^\pm_{s,t}\}}$$

otherwise. Note that (i)-(iv) uniquely define the law of

$$(\varepsilon^+_{s_1,t_1}, U^+_{s_1,t_1}, \varepsilon^-_{s_1,t_1}, U^-_{s_1,t_1}, \cdots, \varepsilon^+_{s_n,t_n}, U^+_{s_n,t_n}, \varepsilon^-_{s_n,t_n}, U^-_{s_n,t_n}, W)$$

for all $s_i < t_i, 1 \leq i \leq n$.

By construction, for all $s < t, u < v$, if $\mathbb{P}(\mathrm{m}^\pm_{s,t} = \mathrm{m}^\pm_{u,v}) > 0$, then

$$\mathbb{P}(\varepsilon^\pm_{s,t} = \varepsilon^\pm_{u,v}, U^\pm_{s,t} = U^\pm_{u,v} | \mathrm{m}^\pm_{s,t} = \mathrm{m}^\pm_{u,v}) = 1. \tag{7}$$

For $s \leq t, x \in \mathbb{R}$, define

$$\tau^\pm_s(x) = \inf\{r \geq s : W_{s,r} = \mp|x|\}$$

and set

$$\begin{aligned}
\varphi^\pm_{s,t}(x) &= (x \pm \mathrm{sgn}(x) W_{s,t}) 1_{\{t \leq \tau^\pm_s(x)\}} + \varepsilon^\pm_{s,t} W^\pm_{s,t} 1_{\{t > \tau^\pm_s(x)\}}, \\
K^\pm_{s,t}(x) &= \delta_{x \pm \mathrm{sgn}(x) W_{s,t}} 1_{\{t \leq \tau^\pm_s(x)\}} + \big(U^\pm_{s,t} \delta_{W^\pm_{s,t}} + (1 - U^\pm_{s,t}) \delta_{-W^\pm_{s,t}}\big) 1_{\{t > \tau^\pm_s(x)\}}.
\end{aligned}$$

Recall the following

Theorem 2. *[7] (i) (φ^+, W) and $(\varphi^-, -W)$ solve Tanaka's equation (1).*

(ii) (K^+, W) and $(K^-, -W)$ solve Tanaka's equation (2).

(iii) For all $x \in \mathbb{R}$, all $s \leq t$ and all bounded continuous function f, a.s.

$$K^\pm_{s,t} f(x) = E[f(\varphi^\pm_{s,t}(x)) | K^\pm].$$

2.2 Modification of flows

For our later needs, we will construct modifications of $\varphi^\pm$ and of $K^\pm$ which are measurable with respect to (s, t, x, ω). On a set of probability 1, define for all $s < t$, $(s_n, t_n) = (\frac{\lfloor ns \rfloor + 1}{n}, \frac{\lfloor nt \rfloor - 1}{n})$ and

$$(\widetilde{\varepsilon}^\pm_{s,t}, \widetilde{U}^\pm_{s,t}) = (\limsup_{n \to \infty} \varepsilon^\pm_{s_n,t_n}, \limsup_{n \to \infty} U^\pm_{s_n,t_n}).$$

Then, we have the following

Lemma 1. *(i) For all $s < t$, a.s. $\widetilde{\varepsilon}^\pm_{s,t} = \varepsilon^\pm_{s,t}$, $\widetilde{U}^\pm_{s,t} = U^\pm_{s,t}$.*

(ii) Consider the random sets

$$\mathscr{D}^+ = \{(s,t) \in \mathbb{R}^2; s < t, m^+_{s,t} < min(W_s, W_t)\},$$

$$\mathscr{D}^- = \{(s,t) \in \mathbb{R}^2; s < t, m^-_{s,t} > max(W_s, W_t)\}.$$

Then a.s. for all (s,t) and (u,v) in $\mathscr{D}^\pm$,

$$m^\pm_{s,t} = m^\pm_{u,v} \Longrightarrow (\widetilde{\varepsilon}^\pm_{s,t}, \widetilde{U}^\pm_{s,t}) = (\widetilde{\varepsilon}^\pm_{u,v}, \widetilde{U}^\pm_{u,v}).$$

Proof. (i) By (7), a.s. for all $s < t, u < v$ such that $(s,t,u,v) \in \mathbb{Q}^4$, we have

$$m^\pm_{s,t} = m^\pm_{u,v} \Longrightarrow (\varepsilon^\pm_{s,t}, U^\pm_{s,t}) = (\varepsilon^\pm_{u,v}, U^\pm_{u,v}).$$

Fix $s < t$. With probability 1, $\mathrm{m}^\pm_{s,t}$ is attained in $]s,t[$ and thus a.s. there exists n_0 such that

$$m^\pm_{s,t} = m^\pm_{s_n,t_n} = m^\pm_{s_{n_0},t_{n_0}} \text{ for all } n \geq n_0. \tag{8}$$

Taking the limit, we get $(\widetilde{\varepsilon}^\pm_{s,t}, \widetilde{U}^\pm_{s,t}) = (\varepsilon^\pm_{s_{n_0},t_{n_0}}, U^\pm_{s_{n_0},t_{n_0}})$ a.s. From (7) and (8), we also have that $(\varepsilon^\pm_{s,t}, U^\pm_{s,t}) = (\varepsilon^\pm_{s_{n_0},t_{n_0}}, U^\pm_{s_{n_0},t_{n_0}})$ a.s. and (i) is proved.

(ii) With probability 1, for all (s,t) and (u,v) in $\mathscr{D}^\pm$, if $\mathrm{m}^\pm_{s,t} = \mathrm{m}^\pm_{u,v}$, then $\exists n_0 : \mathrm{m}^\pm_{s_n,t_n} = \mathrm{m}^\pm_{u_n,v_n}$ for all $n \geq n_0$, which implies that

$$\exists n_0 : (\varepsilon^\pm_{s_n,t_n}, U^\pm_{s_n,t_n}) = (\varepsilon^\pm_{u_n,v_n}, U^\pm_{u_n,v_n}) \text{ for all } n \geq n_0$$

and thus that $\widetilde{\varepsilon}^\pm_{s,t} = \widetilde{\varepsilon}^\pm_{u,v}$ and that $\widetilde{U}^\pm_{s,t} = \widetilde{U}^\pm_{u,v}$. $\square$

We may now consider the following modifications of $\varphi^\pm$ and $K^\pm$ defined for all $s \leq t, x \in \mathbb{R}$ by

$$\begin{aligned}
\widetilde{\varphi}^\pm_{s,t}(x) &= (x \pm \mathrm{sgn}(x) W_{s,t}) 1_{\{t \leq \tau^\pm_s(x)\}} + \widetilde{\varepsilon}^\pm_{s,t} W^\pm_{s,t} 1_{\{t > \tau^\pm_s(x)\}}, \\
\widetilde{K}^\pm_{s,t}(x) &= \delta_{x \pm \mathrm{sgn}(x) W_{s,t}} 1_{\{t \leq \tau^\pm_s(x)\}} + \big(\widetilde{U}^\pm_{s,t} \delta_{W^\pm_{s,t}} + (1 - \widetilde{U}^\pm_{s,t}) \delta_{-W^\pm_{s,t}}\big) 1_{\{t > \tau^\pm_s(x)\}}.
\end{aligned}$$

Then Theorem 2 holds also for $\widetilde{\varphi}^\pm, \widetilde{K}^\pm$ (because (i), (ii), (iii) and (iv) stated at the begining of Section 2.1 are satisfied by $(\widetilde{\varepsilon}^\pm, \widetilde{U}^\pm, W)$).

Lemma 2. *(i) The mapping*

$$(s,t,x,\omega) \longmapsto (\widetilde{\varphi}^\pm_{s,t}(x,\omega), \widetilde{K}^\pm_{s,t}(x,\omega))$$

is measurable from $\{(s,t,x,\omega), s \leq t, x \in \mathbb{R}, \omega \in \Omega\}$ into $\mathbb{R} \times \mathcal{P}(\mathbb{R})$.
(ii) For all s,t,x, a.s.

$$\varphi^\pm_{s,t}(x) = \widetilde{\varphi}^\pm_{s,t}(x) \text{ and } K^\pm_{s,t}(x) = \widetilde{K}^\pm_{s,t}(x).$$

Proof. (i) Clearly

$$(s,t,\omega) \longmapsto (\widetilde{\varepsilon}^\pm_{s,t}(\omega), \widetilde{U}^\pm_{s,t}(\omega), W_{s,t}(\omega))$$

is measurable. For all $t \geq s$, we have

$$\{\tau_s^+(x) > t\} = \{\inf_{s \leq r \leq t} W_{s,r} + |x| > 0\}$$

which shows that $(s, x, \omega) \longmapsto \tau_s^+(x, \omega)$ is measurable and a fortiori $(s, x, \omega) \longmapsto \tau_s^-(x, \omega)$ is also measurable. (ii) is a consequence of Lemma 1 (i). $\square$

To simplify notations, throughout the rest of the paper, we will denote $\widetilde{\varepsilon}_{s,t}^{\pm}, \widetilde{U}_{s,t}^{\pm}, \widetilde{\varphi}_{s,t}^{\pm}, \widetilde{K}_{s,t}^{\pm}$ simply by $\varepsilon_{s,t}^{\pm}, U_{s,t}^{\pm}, \varphi_{s,t}^{\pm}, K_{s,t}^{\pm}$.

2.3 The construction of K^{m^+,m^-}

In this paragraph, we construct a stochastic flow of kernels K^{m^+,m^-} and a stochastic flow of mappings φ respectively from (K^+, K^-) and from (φ^+, φ^-). Let

$$\rho_s = \inf\{r \geq s,\ \sup(W_{s,r}^+, W_{s,r}^-) = l\}. \tag{9}$$

We first define $(\varphi_{s,t})_{s \leq t \leq \rho_s}$. For $t \in [s, \rho_s]$, set

$$\begin{aligned} \varphi_{s,t}(1) &= \exp(i\varphi_{s,t}^+(0)), \\ \varphi_{s,t}(e^{il}) &= \exp(i(l + \varphi_{s,t}^-(0))) \end{aligned}$$

and for $z \in \mathscr{C} \setminus \{1, e^{il}\}$ and $t \in [s, \rho_s]$, set

$$\begin{aligned} \varphi_{s,t}(z) &= z e^{i\epsilon(z) W_{s,t}} 1_{\{t \leq \tau_s(z)\}} \\ &+ \left(\varphi_{s,t}(1) 1_{\{z e^{i\epsilon(z) W_{s,\tau_s(z)}} = 1\}} + \varphi_{s,t}(e^{il}) 1_{\{z e^{i\epsilon(z) W_{s,\tau_s(z)}} = e^{il}\}}\right) 1_{\{t > \tau_s(z)\}}, \end{aligned}$$

where

$$\tau_s(z) = \inf\{r \geq s,\ z e^{i\epsilon(z) W_{s,r}} = 1 \text{ or } e^{il}\}.$$

Note that on $\{\tau_s(z) < \rho_s\} \cap \{z e^{i\epsilon(z) W_{s,\tau_s(z)}} = 1\}$, we have $W_{s,\tau_s(z)}^+ = 0$ and consequently $\varphi_{s,\tau_s(z)}(1) = 1$. Also, on $\{\tau_s(z) < \rho_s\} \cap \{z e^{i\epsilon(z) W_{s,\tau_s(z)}} = e^{il}\}$, we have $W_{s,\tau_s(z)}^- = 0$ and so $\varphi_{s,\tau_s(z)}(e^{il}) = e^{il}$.
Since $(s, \omega) \longmapsto \rho_s(\omega)$ and $(s, z, \omega) \longmapsto \tau_s(z, \omega)$ are measurabe, it follows from Lemma 2 that

$$(s, t, z, \omega) \longmapsto \varphi_{s,t}(z, \omega) 1_{\{s \leq t \leq \rho_s(\omega)\}}$$

is measurable from $\{(s, t, z, \omega), s \leq t, z \in \mathscr{C}, \omega \in \Omega\}$ into $\mathscr{C}$. Now we consider the sequence of stopping times $(\rho_s^k)_{k \geq 0}$ such that $\rho_s^0 = s$ and $\rho_s^{k+1} = \rho_{\rho_s^k}$ for $k \geq 0$.

Define for all $s \le t$,

$$\varphi_{s,t} = \sum_{k\ge 0} 1_{\{\rho_s^k \le t < \rho_s^{k+1}\}} \varphi_{\rho_s^k,t} \circ \varphi_{\rho_s^{k-1},\rho_s^k} \circ \cdots \circ \varphi_{s,\rho_s}.$$

Then $(s,t,z,\omega) \longmapsto \varphi_{s,t}(z,\omega)$ is measurable from $\{(s,t,z,\omega), s \le t, z \in \mathscr{C}, \omega \in \Omega\}$ into $\mathscr{C}$. By the same way, we define $(K_{s,t}^{m^+,m^-})_{s\le t\le \rho_s}$ for $t \in [s,\rho_s]$

$$\begin{aligned} K_{s,t}^{m^+,m^-}(1) &= U_{s,t}^+ \delta_{\exp(iW_{s,t}^+)} + (1-U_{s,t}^+)\delta_{\exp(-iW_{s,t}^+)}, \\ K_{s,t}^{m^+,m^-}(e^{il}) &= U_{s,t}^- \delta_{\exp(i(l+W_{s,t}^-))} + (1-U_{s,t}^-)\delta_{\exp(i(l-W_{s,t}^-))} \end{aligned}$$

and for $z \in \mathscr{C} \setminus \{1, e^{il}\}$ and $t \in [s,\rho_s]$

$$\begin{aligned} K_{s,t}^{m^+,m^-}(z) &= \delta_{ze^{i\epsilon(z)W_{s,t}}} 1_{\{t \le \tau_s(z)\}} \\ &+ \left(K_{s,t}^{m^+,m^-}(1) 1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}}=1\}} + K_{s,t}^{m^+,m^-}(e^{il}) 1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}}=e^{il}\}}\right) 1_{\{t>\tau_s(z)\}}. \end{aligned}$$

Define now for all $s \le t$,

$$K_{s,t}^{m^+,m^-} = \sum_{k\ge 0} 1_{\{\rho_s^k \le t < \rho_s^{k+1}\}} K_{s,\rho_s}^{m^+,m^-} \cdots K_{\rho_s^{k-1},\rho_s^k}^{m^+,m^-} K_{\rho_s^k,t}^{m^+,m^-}.$$

Then $(s,t,z,\omega) \longmapsto K_{s,t}^{m^+,m^-}(z,\omega)$ is measurable from $\{(s,t,z,\omega), s \le t, z \in \mathscr{C}, \omega \in \Omega\}$ into $\mathcal{P}(\mathscr{C})$. For every choice $s_1 < t_1 < \cdots < s_n < t_n$, $(\varphi_{s_i,t_i}, K_{s_i,t_i}^{m^+,m^-})$ is $\sigma(\varepsilon_{u,v}^+, \varepsilon_{u,v}^-, U_{u,v}^+, U_{u,v}^-, W_{u,v}, s_i \le u \le v \le t_i)$ measurable and these σ-fields are independent for $1 \le i \le n$ by construction. This implies the independence of the family $\{(\varphi_{s_i,t_i}, K_{s_i,t_i}^{m^+,m^-}),\ 1 \le i \le n\}$. It is also clear that the laws of $\varphi_{s,t}$ and $K_{s,t}^{m^+,m^-}$ only depend on $t-s$.

2.4 The flow property for K^{m^+,m^-} and φ.

To prove the flow property for both φ and K^{m^+,m^-}, we start by the following

Proposition 3. *Let S, T be two finite $(\mathcal{F}_{-\infty,r}^W)_{r\in\mathbb{R}}$-stopping times such that $S \le T \le \rho_S$. Then a.s. for all $u \in [T,\rho_S], z \in \mathscr{C}$, we have*

$$\varphi_{S,u}(z) = \varphi_{T,u} \circ \varphi_{S,T}(z)$$

and

$$K_{S,u}^{m^+,m^-}(z) = K_{S,T}^{m^+,m^-} K_{T,u}^{m^+,m^-}(z).$$

Proof. Define

$$\begin{aligned} \Omega_1 &= \{\omega \in \Omega : \forall\ (s_1,t_1),(s_2,t_2) \in \mathscr{D}^\pm, m_{s_1,t_1}^\pm = m_{s_2,t_2}^\pm \Rightarrow \varepsilon_{s_1,t_1}^\pm = \varepsilon_{s_2,t_2}^\pm\} \\ \Omega_2 &= \{\omega \in \Omega : m_{T,T+r}^+ < W_T < m_{T,T+r}^-, m_{S,S+r}^+ < W_S < m_{S,S+r}^- \text{ for all } r > 0\}. \end{aligned}$$

Then $\mathbb{P}(\Omega_1) = 1$ (see Lemma 1 (ii)). It is also known that $\mathbb{P}(\Omega_2) = 1$ (see [4] page 94). We will prove the proposition on the set of probability 1: $\tilde{\Omega} = \Omega_1 \cap \Omega_2$ and we first prove the result for φ. From now on, we fix $\omega \in \tilde{\Omega}$. Define

$$\begin{aligned}
E_{(i)} &= \{(u,z) : T \le u \le \rho_S, u < \tau_S(z)\},\\
E_{(ii)} &= \{(u,z) : T < \tau_S(z) \le u \le \rho_S\},\\
E_{(iii)} &= \{(u,z) : \tau_S(z) \le T \le u \le \rho_S, u < \tau_T(\varphi_{S,T}(z))\},\\
E_{(iv)} &= \{(u,z) : \tau_S(z) \le T \le \tau_T(\varphi_{S,T}(z)) \le u \le \rho_S\}.
\end{aligned}$$

Then $E_{(i)} \cup E_{(ii)} \cup E_{(iii)} \cup E_{(iv)} = [T, \rho_S] \times \mathscr{C}$. For all $z \in \mathscr{C}$, set $Z = \varphi_{S,T}(z)$ and $\theta = \arg(z)$.

(i) Let $(z,u) \in E_{(i)}$. Then as $T < \tau_S(z)$, we have $\theta \notin \{0, l\}, Z = ze^{i\epsilon(z)W_{S,T}}$ and

$$\begin{aligned}
\tau_T(Z) &= \inf\{r \ge T,\ Ze^{i\epsilon(Z)W_{T,r}} = 1 \text{ or } e^{il}\}\\
&= \inf\{r \ge T,\ ze^{i(\epsilon(z)W_{S,T}+\epsilon(Z)W_{T,r})} = 1 \text{ or } e^{il}\} = \tau_S(z)
\end{aligned}$$

since $\epsilon(z) = \epsilon(Z)$. Therefore $u < \tau_T(Z)$ and $\varphi_{T,u} \circ \varphi_{S,T}(z) = Ze^{i\epsilon(Z)W_{T,u}} = ze^{i\epsilon(z)W_{S,u}} = \varphi_{S,u}(z)$.

(ii) Let $(z,u) \in E_{(ii)}$. Then, we still have $\tau_T(Z) = \tau_S(z)$ and $\varphi_{T,\tau_T(Z)}(Z) = \varphi_{S,\tau_S(z)}(z)$.

Recall that

$$\varphi_{S,u}(z) = \varphi_{S,u}(1)1_{\{\varphi_{S,\tau_S(z)}(z)=1\}} + \varphi_{S,u}(e^{il})1_{\{\varphi_{S,\tau_S(z)}(z)=e^{il}\}}$$

and

$$\varphi_{T,u}(Z) = \varphi_{T,u}(1)1_{\{\varphi_{T,\tau_T(Z)}(Z)=1\}} + \varphi_{T,u}(e^{il})1_{\{\varphi_{T,\tau_T(Z)}(Z)=e^{il}\}}.$$

Suppose for example $\varphi_{S,\tau_S(z)}(z) = \varphi_{T,\tau_T(Z)}(Z) = 1$, then $W^+_{T,\tau_T(Z)} = W^+_{S,\tau_S(z)} = 0$ and so $W^+_{T,r} = W^+_{S,r}$ (and a fortiori $m^+_{T,r} = m^+_{S,r}$) for all $r \ge \tau_T(Z)(= \tau_S(z))$. From the definition,

$$\varphi_{S,u}(z) = \varphi_{S,u}(1) = \exp(i\varphi^+_{S,u}(0)) \text{ and } \ \varphi_{T,u}(Z) = \varphi_{T,u}(1) = \exp(i\varphi^+_{T,u}(0)).$$

If $W^+_{T,u} = W^+_{S,u} = 0$, then $\varphi_{S,u}(z) = \varphi_{T,u}(Z) = 1$. Suppose that $W^+_{T,u} = W^+_{S,u} > 0$, then $W_u > \mathrm{m}^+_{T,u}$ and $W_u > \mathrm{m}^+_{S,u}$. Since $\omega \in \Omega_2$, we have

$$W_T > \mathrm{m}^+_{T,u} \text{ and } W_S > \mathrm{m}^+_{S,u}.$$

In other words, (T,u) and (S,u) are in $\mathscr{D}^+$ so that $\varepsilon^+_{S,u} = \varepsilon^+_{T,u}$ and $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$.

(iii) Let $(z,u) \in E_{(iii)}$. Assume for example that $\varphi_{S,\tau_S(z)}(z) = 1$, then $Z = \varphi_{S,T}(1) = e^{i\varphi^+_{S,T}(0)}$ since $T \le \rho_S$ and

$$\begin{aligned}
\varphi_{T,u}(Z) &= \exp(i(\varphi^+_{S,T}(0) + \epsilon(Z)W_{T,u}))\\
&= \exp(i(\varepsilon^+_{S,T}W^+_{S,T} + \epsilon(Z)W_{T,u})).
\end{aligned}$$

As $T \le u < \tau_T(Z)$, it follows that $Z \notin \{1, e^{il}\}$ (if $Z \in \{1, e^{il}\}$, then $\tau_T(Z) = T$), $\epsilon(Z) = \varepsilon^+_{S,T}$ and so $\varphi_{T,u}(Z) = Z\exp(i\varepsilon^+_{S,T}W_{T,u}) = \exp(i\varepsilon^+_{S,T}(W_u - \mathrm{m}^+_{S,T}))$. As $Z \neq 1$, we necessarily have $W^+_{S,T} > 0$. Thus if $\varepsilon^+_{S,T} = 1$,

$$\tau_T(Z) = \inf\{r \ge T : \ W_r - \mathrm{m}^+_{S,T} = 0 \text{ or } \ l\}$$

and if $\varepsilon^+_{S,T} = -1$,

$$\tau_T(Z) = \inf\{r \ge T : \ W_r - \mathrm{m}^+_{S,T} = 0 \text{ or } \ 2\pi - l\}.$$

Since $u < \tau_T(Z)$, we have $\mathrm{m}^+_{S,u} = \mathrm{m}^+_{S,T}$ and $\varphi_{T,u}(Z) = \exp(i\varepsilon^+_{S,T}W^+_{S,u})$. On the other hand, since $u \le \rho_S$,

$$\varphi_{S,u}(z) = \exp(i\varphi^+_{S,u}(0)) = \exp(i\varepsilon^+_{S,u}W^+_{S,u}).$$

But $(S,T) \in \mathscr{D}^+$ (from $W^+_{S,T} > 0$), $(S,u) \in \mathscr{D}^+$ (from $u < \tau_T(Z)$ which entails that $W^+_{S,u} > 0$). Consequently $\varepsilon^+_{S,u} = \varepsilon^+_{S,T}$ and so $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$. The case $\varphi_{S,\tau_S(z)}(z) = e^{il}$ can be done similarly.
(iv) Let $(z,u) \in E_{(iv)}$. Assume for example that $\varphi_{S,\tau_S(z)}(z) = 1$ so that $W^+_{S,\tau_S(z)} = 0$. Consider the first case: $\varepsilon^+_{S,T} = 1$. Then $Z = e^{iW^+_{S,T}}$ and

$$\tau_T(Z) = \inf\{r \ge T : \ W_r - \mathrm{m}^+_{S,T} \in \{0, l\}\}.$$

If $W_{\tau_T(Z)} - \mathrm{m}^+_{S,T} = l$, then $u = \tau_T(Z) = \rho_S$ and $\varphi_{S,u}(z) = \varphi_{T,u}(Z) = e^{il}$.
If $W_{\tau_T(Z)} - \mathrm{m}^+_{S,T} = 0$, then $\varphi_{T,\tau_T(Z)}(Z) = 1$ and $\varphi_{T,u}(Z) = \varphi_{T,u}(1)$.
Since $\varphi_{S,\tau_S(z)}(z) = 1$, we have $\varphi_{S,u}(z) = \varphi_{S,u}(1)$. Moreover $W^+_{T,\tau_T(Z)} = W^+_{S,\tau_T(Z)} = 0$, which implies $W^+_{T,u} = W^+_{S,u}$ (since $u \ge \tau_T(Z)$).
Now, if u satisfies $W^+_{T,u} = W^+_{S,u} = 0$, then $\varphi_{T,u}(Z) = \varphi_{S,u}(z) = 1$. If not, $\mathrm{m}^+_{T,u} = \mathrm{m}^+_{S,u}$ and $(T,u), (S,u)$ are in $\mathscr{D}^+$. This implies $\varepsilon^+_{T,u} = \varepsilon^+_{S,u}$ and $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$ exactly as in (ii).
Assume now that $\varepsilon^+_{S,T} = -1$, then $\tau_T(Z)$ satisfies $W_{\tau_T(Z)} - \mathrm{m}^+_{S,T} = 0$ (recall that $\tau_T(Z) \le \rho_S$) and $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$ as before.
The result for K^{m^+,m^-} can be proved by replacing $\varphi_{S,T}(z)$ by $e^{iW^+_{S,T}}$ in $E_{(iii)}$ and $E_{(iv)}$. However, the proof remains similar. □

Corollary 2. *Let $S \le T$ be two finite $(\mathcal{F}^W_{-\infty,r})_{r\in\mathbb{R}}$-stopping times. Then, with probability 1, for all $u \ge T, z \in \mathscr{C}$, we have*

$$\varphi_{S,u}(z) = \varphi_{T,u} \circ \varphi_{S,T}(z)$$

and

$$K^{m^+,m^-}_{S,u}(z) = K^{m^+,m^-}_{S,T}K^{m^+,m^-}_{T,u}(z).$$

Proof. Fix $k \in \mathbb{N}$ and define the family of $(\mathcal{F}^W_{-\infty,r})_{r\in\mathbb{R}}$-stopping times $(T^i)_{i\geq 0}$ by $T^0 = (T \vee \rho^k_S) \wedge \rho^{k+1}_S$ and $T^i = \rho_{T^{i-1}}$ for $i \geq 1$. As $r \longmapsto \rho_r$ is increasing, we have $\rho^{k+i}_S \leq T^i \leq \rho^{k+i+1}_S$ for all $i \geq 0$. Applying successively Proposition 3, we have a.s. for all $z \in \mathscr{C}$ and all $i \geq 0$,

$$\varphi_{S,u}(z) = \varphi_{\rho^{k+i}_S,u} \circ \varphi_{T^{i-1},\rho^{k+i}_S} \circ \cdots \circ \varphi_{T^0,\rho^{k+1}_S} \circ \varphi_{\rho^k_S,T^0} \circ \varphi_{S,\rho^k_S}(z) \text{ for all } u \in [\rho^{k+i}_S, T^i]$$

and

$$\varphi_{S,u}(z) = \varphi_{T^i,u} \circ \varphi_{\rho^{k+i}_S,T^i} \circ \cdots \circ \varphi_{T^0,\rho^{k+1}_S} \circ \varphi_{\rho^k_S,T^0} \circ \varphi_{S,\rho^k_S}(z) \text{ for all } u \in [T^i, \rho^{k+i+1}_S].$$

On $\{\rho^k_S \leq T < \rho^{k+1}_S\}$, we have $T^i = \rho^i_T$ for all $i \geq 0$ whence a.s. on $\{\rho^k_S \leq T < \rho^{k+1}_S\}$, for all $z \in \mathscr{C}$ and all $i \geq 0$,

$$\varphi_{S,u}(z) = \varphi_{\rho^{k+i}_S,u} \circ \varphi_{\rho^{i-1}_T,\rho^{k+i}_S} \circ \cdots \circ \varphi_{T,\rho^{k+1}_S} \circ \varphi_{S,T}(z) \text{ for all } u \in [\rho^{k+i}_S, \rho^i_T]$$

and

$$\varphi_{S,u}(z) = \varphi_{\rho^i_T,u} \circ \varphi_{\rho^{k+i}_S,\rho^i_T} \circ \cdots \circ \varphi_{T,\rho^{k+1}_S} \circ \varphi_{S,T}(z) \text{ for all } u \in [\rho^i_T, \rho^{k+i+1}_S].$$

Now define the family $(S^i)_{i\geq 1}$ of $(\mathcal{F}^W_{-\infty,r})_{r\in\mathbb{R}}$-stopping times by $S^1 = (T \vee \rho^{k+1}_S) \wedge \rho^1_T$ and $S^{i+1} = \rho_{S^i}$ for $i \geq 1$. Then for all $i \geq 0$, $\rho^i_T \leq S^{i+1} \leq \rho^{i+1}_T$. Applying again successively Proposition 3, we get a.s. for all $z \in \mathscr{C}$ and all $i \geq 0$,

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{\rho^i_T,u} \circ \varphi_{S^i,\rho^i_T} \circ \cdots \circ \varphi_{S^1,\rho^1_T} \circ \varphi_{T,S^1}(\varphi_{S,T}(z)) \text{ for all } u \in [\rho^i_T, S^{i+1}]$$

and

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{S^{i+1},u} \circ \varphi_{\rho^i_T,S^{i+1}} \circ \cdots \circ \varphi_{S^1,\rho^1_T} \circ \varphi_{T,S^1}(\varphi_{S,T}(z)) \text{ for all } u \in [S^{i+1}, \rho^{i+1}_T].$$

On $\{\rho^k_S \leq T < \rho^{k+1}_S\}$, we have $S^i = \rho^{k+i}_S$ for all $i \geq 1$ and consequently a.s. on $\{\rho^k_S \leq T < \rho^{k+1}_S\}$, for all $z \in \mathscr{C}$ and all $i \geq 0$,

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{\rho^i_T,u} \circ \varphi_{\rho^{k+i}_S,\rho^i_T} \circ \cdots \circ \varphi_{\rho^{k+1}_S,\rho^1_T} \circ \varphi_{T,\rho^{k+1}_S}(\varphi_{S,T}(z)) \text{ for all } u \in [\rho^i_T, \rho^{k+i+1}_S]$$

and

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{\rho^{k+i+1}_S,u} \circ \varphi_{\rho^i_T,\rho^{k+i+1}_S} \circ \cdots \circ \varphi_{\rho^{k+1}_S,\rho^1_T} \circ \varphi_{T,\rho^{k+1}_S}(\varphi_{S,T}(z)) \text{ for all } u \in [\rho^{k+i+1}_S, \rho^{i+1}_T].$$

We have thus shown that a.s. for all $z \in \mathscr{C}$ and all $u \geq T$,

$$1_{\{\rho^k_S \leq T < \rho^{k+1}_S\}} \varphi_{T,u} \circ \varphi_{S,T}(z) = 1_{\{\rho^k_S \leq T < \rho^{k+1}_S\}} \varphi_{S,u}(z).$$

By summing over k, we get that a.s. $\forall z \in \mathscr{C}, \forall u \geq T$, $\varphi_{T,u} \circ \varphi_{S,T}(z) = \varphi_{S,u}(z)$. The flow property for K^{m^+,m^-} holds by the same reasoning. $\square$

2.5 K^{m^+,m^-} can be obtained by filtering φ

For all $-\infty \le s \le t \le +\infty$, let

$$\mathcal{F}_{s,t}^{U^+,U^-,W} = \sigma(U_{u,v}^+, U_{u,v}^-, W_{u,v}; s \le u \le v \le t) = \sigma(K_{u,v}^+, K_{u,v}^-; s \le u \le v \le t).$$

Corollary 2 entails the following

Proposition 4. *For all $z \in \mathscr{C}$, all $s < t$ and all continuous function f, a.s.*

$$K_{s,t}^{m^+,m^-} f(z) = E\left[f(\varphi_{s,t}(z)) | \mathcal{F}_{s,t}^{U^+,U^-,W}\right].$$

Proof. Fix $s \le t, z \in \mathscr{C}$ and $f \in C(\mathscr{C})$. Properties (ii) and (iii) of Section 2.1 imply that a.s.

$$K_{s,t}^{m^+,m^-} f(z) 1_{\{s \le t \le \rho_s\}} = E\left[f(\varphi_{s,t}(z)) | \mathcal{F}_{s,t}^{U^+,U^-,W}\right] 1_{\{s \le t \le \rho_s\}}.$$

Define

$$\mathcal{F}_{s,t}^{\varepsilon^+,\varepsilon^-,U^+,U^-,W} = \sigma(\varepsilon_{u,v}^\pm, U_{u,v}^\pm, W_{u,v}; s \le u \le v \le t) = \sigma(\varphi_{u,v}^\pm, K_{u,v}^\pm; s \le u \le v \le t).$$

If Z is a random variable independent of $\mathcal{F}_{s,t}^{\varepsilon^+,\varepsilon^-,U^+,U^-,W}$, then a.s.

$$K_{s,t}^{m^+,m^-} f(Z) 1_{\{s \le t \le \rho_s\}} = E\left[f(\varphi_{s,t}(Z)) | \mathcal{F}_{s,t}^{U^+,U^-,W}\right] 1_{\{s \le t \le \rho_s\}}. \tag{10}$$

For $n \ge 1$ and $i \in [0, n]$, let $t_i^n = s + \frac{(t-s)i}{n}$, $A_{n,i} = \{t_i^n \le \rho_{t_{i-1}^n}\}$ and for $n \ge 1$ let $A_n = \cap_{i=1}^n A_{n,i}$. Note that $A_{n,i} \in \mathcal{F}_{t_{i-1}^n,t_i^n}^W$ and $A_n \in \mathcal{F}_{s,t}^W$. Then since $K^\pm$ and $\varphi^\pm$ are stochastic flows, $\mathcal{F}_{s,t}^{\varepsilon^+,\varepsilon^-,U^+,U^-,W} = \bigvee_{i=1}^n F_{t_{i-1}^n,t_i^n}^{\varepsilon^+,\varepsilon^-,U^+,U^-,W}$. By Corollary 2, a.s.

$$K_{s,t}^{m^+,m^-}(z) = K_{s,t_1^n}^{m^+,m^-} \cdots K_{t_{n-1}^n,t}^{m^+,m^-}(z)$$

and

$$\varphi_{s,t}(z) = \varphi_{t_{n-1}^n,t} \circ \cdots \circ \varphi_{s,t_1^n}(z).$$

Recall that the σ-fields $\left(\mathcal{F}_{t_{i-1}^n,t_i^n}^{\varepsilon^+,\varepsilon^-,U^+,U^-,W}\right)_{1 \le i \le n}$ are independent. Then, using (10), we get that a.s.

$$K_{s,t}^{m^+,m^-} f(z) 1_{A_n} = E\left[f(\varphi_{s,t}(z)) | \mathcal{F}_{s,t}^{U^+,U^-,W}\right] 1_{A_n},$$

and therefore a.s.

$$K_{s,t}^{m^+,m^-} f(z) = E\left[f(\varphi_{s,t}(z)) | \mathcal{F}_{s,t}^{U^+,U^-,W}\right] 1_{A_n} + \left(K_{s,t}^{m^+,m^-} f(z) - E\left[f(\varphi_{s,t}(z)) | \mathcal{F}_{s,t}^{U^+,U^-,W}\right]\right) 1_{A_n^c}.$$

To finish the proof, it remains to prove that $\mathbb{P}(A_n^c) \to 0$ as $n \to \infty$. Write

$$\mathbb{P}(A_n^c) \le \sum_{i=1}^{n} \mathbb{P}(A_{n,i}^c) = \sum_{i=1}^{n} \mathbb{P}(t_i^n - t_{i-1}^n > \rho_{t_{i-1}^n} - t_{i-1}^n) = n\mathbb{P}\left(\frac{t-s}{n} > \rho_0\right).$$

Let $\rho^{\pm} = \inf\{r \ge 0 : W_{0,r}^{\pm} = l\}$. Then

$$\mathbb{P}(A_n^c) \le n\left(\mathbb{P}\left(\frac{t-s}{n} > \rho^+\right) + \mathbb{P}\left(\frac{t-s}{n} > \rho^-\right)\right) = 2n\mathbb{P}\left(\frac{t-s}{n} > \rho^+\right).$$

We have $\rho^+ \overset{law}{=} \inf\{r \ge 0 : |W_r| = l\}$. Let $T_l = \inf\{r \ge 0 : W_r = l\}$, then

$$\mathbb{P}(A_n^c) \le 4n\mathbb{P}\left(\frac{t-s}{n} > T_l\right) = 4n\int_{\frac{t-s}{n}}^{+\infty} \frac{l}{\sqrt{2\pi x^3}} \exp(\frac{-l^2}{2x})dx$$

(see [4] page 80). By the change of variable $v = nx$, the right hand side converges to 0 as $n \to \infty$ which finishes the proof. □

2.6 The L^2 continuity

To conclude that K^{m^+,m^-} and φ are two stochastic flows, it remains to prove the following

Proposition 5. *For all $t \ge 0$, $\theta \in [0, 2\pi[$ and $f \in C(\mathscr{C})$, we have*

$$\lim_{z \to e^{i\theta}} E\left[\left(f(\varphi_{0,t}(z)) - f(\varphi_{0,t}(e^{i\theta}))\right)^2\right] = \lim_{z \to e^{i\theta}} E\left[\left(K_{0,t}^{m^+,m^-}f(z) - K_{0,t}^{m^+,m^-}f(e^{i\theta})\right)^2\right] = 0.$$

Proof. By Jensen's inequality and Proposition 4, it suffices to prove the result only for φ and by the proof of Lemma 1.11 [6] (see also Lemma 1 [2]), this amounts to show that for all $t > 0, \eta > 0$ and $\theta \in [0, 2\pi[$:

$$\lim_{z \to e^{i\theta}} \mathbb{P}\left(d(\varphi_{0,t}(z), \varphi_{0,t}(e^{i\theta})) > \eta\right) = 0. \tag{11}$$

Fix $\eta > 0, t > 0$ and $\theta \in [0, 2\pi[$. For $z \in \mathscr{C}$, set

$$A_z = \{d(\varphi_{0,t}(z), \varphi_{0,t}(e^{i\theta})) > \eta\}.$$

For simplicity, we will write $\tau(z)$ and φ_t instead of $\tau_0(z)$ and $\varphi_{0,t}$.

First case : $\theta = 0$. For $\alpha \in]0, l[$, we have $\tau(e^{i\alpha}) = \inf\{t \ge 0 : \alpha + W_t = 0 \text{ or } l\}$ and

$$\mathbb{P}(A_{e^{i\alpha}}) \le \mathbb{P}(t < \tau(e^{i\alpha})) + \mathbb{P}\left(A_{e^{i\alpha}} \cap \{\varphi_{\tau(e^{i\alpha})}(e^{i\alpha}) = 1, t \ge \tau(e^{i\alpha})\}\right) + \mathbb{P}(\varphi_{\tau(e^{i\alpha})}(e^{i\alpha}) = e^{il}).$$

If $t \ge \tau(e^{i\alpha})$ and $\varphi_{\tau(e^{i\alpha})}(e^{i\alpha}) = 1$, then $\varphi_t(e^{i\alpha}) = \varphi_t(1)$. Thus

$$\mathbb{P}\left(A_{e^{i\alpha}} \cap \{\varphi_{\tau(e^{i\alpha})}(e^{i\alpha}) = 1, t \ge \tau(e^{i\alpha})\}\right) = 0.$$

From $\lim_{\alpha\to 0+}\tau(e^{i\alpha})=0$ a.s. and

$$\mathbb{P}(\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=e^{il})=\mathbb{P}(\alpha+W_{\tau(e^{i\alpha})}=l),$$

we get $\lim_{\alpha\to 0+}\mathbb{P}(A_{e^{i\alpha}})=0$ and similarly, we can prove that $\lim_{\alpha\to(2\pi)-}\mathbb{P}(A_{e^{i\alpha}})=0$. Thus (11) holds for $\theta=0$ and by the same way for $\theta=l$.

Second case : $\theta\in]l,2\pi[$. For all $\alpha\in]l,2\pi[$, we have

$$\begin{aligned}\mathbb{P}(A_{e^{i\alpha}}) &\leq \mathbb{P}(A_{e^{i\alpha}}\cap\{\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=\varphi_{\tau(e^{i\theta})}(e^{i\theta})=1\})\\ &+ \mathbb{P}(A_{e^{i\alpha}}\cap\{\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=\varphi_{\tau(e^{i\theta})}(e^{i\theta})=e^{il}\})+\epsilon_{\alpha,\theta}\end{aligned}$$

where

$$\epsilon_{\alpha,\theta}=\mathbb{P}(\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=1,\varphi_{\tau(e^{i\theta})}(e^{i\theta})=e^{il})+\mathbb{P}(\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=e^{il},\varphi_{\tau(e^{i\theta})}(e^{i\theta})=1)$$

which converges to 0 as $\alpha\to\theta$. Let us prove that

$$\lim_{\alpha\to\theta}\mathbb{P}(B_\alpha)=0 \text{ where } B_\alpha=A_{e^{i\alpha}}\cap\{\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=\varphi_{\tau(e^{i\theta})}(e^{i\theta})=1\}.$$

For $l<\alpha<\theta$, write

$$\mathbb{P}(B_\alpha)=\mathbb{P}(B_\alpha\cap\{t\leq\tau(e^{i\theta})\})+\mathbb{P}(B_\alpha\cap\{\tau(e^{i\theta})<t<\tau(e^{i\alpha})\})+\mathbb{P}(B_\alpha\cap\{t\geq\tau(e^{i\alpha})\}).$$

Since $\varphi_.(e^{i\alpha})$ and $\varphi_.(e^{i\theta})$ move parallely until one of them hits 1 or e^{il}, it is easy to see that

$$\lim_{\alpha\to\theta-}\Big(\mathbb{P}(B_\alpha\cap\{t\leq\tau(e^{i\theta})\})+\mathbb{P}(B_\alpha\cap\{\tau(e^{i\theta})<t<\tau(e^{i\alpha})\})\Big)=0.$$

Now

$$\begin{aligned}\mathbb{P}(B_\alpha\cap\{t\geq\tau(e^{i\alpha})\}) &= \mathbb{P}(B_\alpha\cap\{\tau(e^{i\alpha})\leq t\wedge\rho_{\tau(e^{i\theta})}\})+\mathbb{P}(B_\alpha\cap\{\rho_{\tau(e^{i\theta})}<\tau(e^{i\alpha})\leq t\})\\ &\leq \mathbb{P}(B_\alpha\cap\{\tau(e^{i\alpha})\leq t\wedge\rho_{\tau(e^{i\theta})}\})+\mathbb{P}(\rho_{\tau(e^{i\theta})}<\tau(e^{i\alpha})).\end{aligned}$$

Obviously $\lim_{\alpha\to\theta}\mathbb{P}(\rho_{\tau(e^{i\theta})}<\tau(e^{i\alpha}))=\mathbb{P}(\rho_{\tau(e^{i\theta})}<\tau(e^{i\theta}))=0$. Set $Y=\varphi_{\tau(e^{i\theta})}(e^{i\alpha})$, then a.s. on $B_\alpha\cap\{\tau(e^{i\alpha})\leq t\wedge\rho_{\tau(e^{i\theta})}\}$, we have $\varphi_t(e^{i\alpha})=\varphi_{\tau(e^{i\theta}),t}(Y)$ by Corollary 2 and $\tau_{\tau(e^{i\theta})}(Y)=\tau(e^{i\alpha})\leq\rho_{\tau(e^{i\theta})}$. We recall that $\varphi_{\tau(e^{i\theta}),s}(Y):=\varphi_{\tau(e^{i\theta}),s}(1)$ for all $s\in[\tau_{\tau(e^{i\theta})}(Y),\rho_{\tau(e^{i\theta})}]$ and thus $\varphi_{\tau(e^{i\theta}),s}(Y)=\varphi_{\tau(e^{i\theta}),s}(1)$ for all $s\geq\tau_{\tau(e^{i\theta})}(Y)$ (by the definition of φ). This shows that a.s. on $B_\alpha\cap\{\tau(e^{i\alpha})\leq t\wedge\rho_{\tau(e^{i\theta})}\}$, we have

$$d(\varphi_t(e^{i\theta}),\varphi_t(e^{i\alpha}))=d(\varphi_{\tau(e^{i\theta}),t}(1),\varphi_{\tau(e^{i\theta}),t}(1))=0$$

Finally $\lim_{\alpha\to\theta-}\mathbb{P}(B_\alpha)=0$ and by interchanging the roles of θ and α, we have $\lim_{\alpha\to\theta+}\mathbb{P}(B_\alpha)=0$. Similarly

$$\lim_{\theta\to\theta}\mathbb{P}\left(A_{e^{i\alpha}}\cap\{\varphi_{\tau(e^{i\theta})}(e^{i\theta})=\varphi_{\tau(e^{i\alpha})}(e^{i\alpha})=e^{il}\}\right)=0$$

so that (11) is satisfied for all θ such that $\theta\in]l,2\pi[$. By the same way, it is also satisfied for all $\theta\in]0,l[$. □

2.7 The flows φ and K^{m^+,m^-} solve $(T_{\mathscr{C}})$

In this paragraph we prove the following

Proposition 6. *Both φ and K^{m^+,m^-} solve $(T_{\mathscr{C}})$.*

Proof. First we check the result for φ. For all $q\geq 1$ and $t>0$ set

$$S_q=\frac{\lfloor qS\rfloor+1}{q},\ S_{q,t}=S_q-\frac{2}{q}+t.$$

First step. Let S be a finite $(\mathcal{F}^W_{0,\cdot})$-stopping time. Then for all $z\in\mathscr{C}$, $f\in C^2(\mathscr{C})$, a.s. $\forall t\in[0,\rho_S-S]$,

$$f(\varphi_{S,S+t}(z))=f(z)+\int_0^t(f'\epsilon)(\varphi_{S,S+u}(z))dW_{S,S+u}+\frac{1}{2}\int_0^t f''(\varphi_{S,S+u}(z))du.$$

We first prove this for $z=1$ and first check that $(\varphi^+_{S,S+t}(0),t\geq 0)$ is a Brownian motion. Let $t>0$, then a.s. $(S,S+t)\in\mathscr{D}^+$ and for q large enough, we have $(S_q,S_{q,t})\in\mathscr{D}^+$ and $\mathrm{m}^+_{S_q,S_{q,t}}=\mathrm{m}^+_{S,S+t}$. Lemma 1 (ii) implies that a.s. for q large enough

$$\varepsilon^+_{S,S+t}=\varepsilon^+_{S_q,S_{q,t}}.$$

Thus a.s.

$$\varphi^+_{S,S+t}(0)=\lim_{q\to\infty}\varphi^+_{S_q,S_{q,t}}(0). \tag{12}$$

Let $0<t_1<\cdots<t_n$ and take a family $(f_i)_{1\leq i\leq n}$ of bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}$.

Using the independence of increments and the stationarity of φ^+, we have

$$
\begin{aligned}
E\left[\prod_{i=1}^n f_i(\varphi^+_{S,S+t_i}(0))\right] &= \lim_{q\to\infty} E\left[\prod_{i=1}^n f_i(\varphi^+_{S_q,S_{q,t_i}}(0))\right]\\
&= \lim_{q\to\infty}\sum_{h\in\mathbb{N}} E\left[\prod_{i=1}^n f_i(\varphi^+_{\frac{h+1}{q},\frac{h-1}{q}+t_i}(0))1_{\{\frac{h}{q}\le S<\frac{h+1}{q}\}}\right]\\
&= \lim_{q\to\infty}\sum_{h\in\mathbb{N}} E\left[\prod_{i=1}^n f_i(\varphi^+_{\frac{h+1}{q},\frac{h-1}{q}+t_i}(0))\right]\mathbb{P}\left(\frac{h}{q}\le S<\frac{h+1}{q}\right)\\
&= \lim_{q\to\infty}\sum_{h\in\mathbb{N}} E\left[\prod_{i=1}^n f_i(\varphi^+_{0,t_i-\frac{2}{q}}(0))\right]\mathbb{P}\left(\frac{h}{q}\le S<\frac{h+1}{q}\right)\\
&= \lim_{q\to\infty} E\left[\prod_{i=1}^n f_i(\varphi^+_{0,t_i-\frac{2}{q}}(0))\right]\\
&= E\left[\prod_{i=1}^n f_i(\varphi^+_{0,t_i}(0))\right].
\end{aligned}
$$

Since $\varphi^+_{0,\cdot}(0)$ is a Brownian motion, the same holds for $\varphi^+_{S,S+\cdot}(0)$. By Itô's formula, we have for all $f\in C^2(\mathscr{C})$ a.s. $\forall t\ge 0$,

$$f(\exp(i\varphi^+_{S,S+t}(0))) = f(1)+\int_0^t f'(\exp(i\varphi^+_{S,S+u}(0)))d\varphi^+_{S,S+u}(0)+\frac{1}{2}\int_0^t f''(\exp(i\varphi^+_{S,S+u}(0)))du.$$

Tanaka's formula for local time yields a.s. $\forall t\in[0,\rho_S-S]$,

$$|\varphi^+_{S,S+t}(0)| = \int_0^t \mathrm{sgn}(\varphi^+_{S,S+u}(0))d\varphi^+_{S,S+u}(0)+L_t$$

where L_t is the local time in 0 of $\varphi^+_{S,S+\cdot}(0)$. Since $|\varphi^+_{S,S+t}(0)| = W^+_{S,S+t}$ for all t, we have a.s. $\forall t\in[0,\rho_S-S]$,

$$\int_0^t \mathrm{sgn}(\varphi^+_{S,S+u}(0))d\varphi^+_{S,S+u}(0) = W_{S,S+t}.$$

Since $\mathrm{sgn}(\varphi^+_{S,S+u}(0)) = \varepsilon^+_{S,S+u}$ a.s., we get a.s. $\forall t\in[0,\rho_S-S]$,

$$\varphi^+_{S,S+t}(0) = \int_0^t \varepsilon^+_{S,S+u}dW_{S,S+u} = \int_0^t \epsilon(\varphi_{S,S+u}(1))dW_{S,S+u}.$$

Recall that $\varphi_{S,S+t}(1) = e^{i\varphi^+_{S,S+t}(0)}$ for all $t\in[0,\rho_S-S]$, thus the first step holds for $z=1$. The first step is similarly satisfied for $z=e^{il}$ and for all $z\in\mathscr{C}\setminus\{1,e^{il}\}$ by distinguishing the cases $t\le\tau_S(z)-S$ and $t>\tau_S(z)-S$.

Second step. Let S be a finite $(\mathcal{F}^W_{0,\cdot})$-stopping time, $\mathcal{G}_t=\sigma(\varphi_{0,u}(z), z\in\mathscr{C}, 0\le u\le t)$, $t\ge 0$. Then $\sigma(\varphi_{S,(S+u)\wedge\rho_S}(z), z\in\mathscr{C}, u\ge 0)$ is independent of $\mathcal{G}_S$.

Clearly

$$\sigma(\varphi_{S,(S+u)\wedge\rho_S}(z), z\in\mathscr{C}, u\ge 0)\subset\sigma(\varphi^+_{S,S+u}(0), u\ge 0)\vee\sigma(\varphi^-_{S,S+u}(0), u\ge 0).$$

Fix $0 < u_1 < \cdots < u_n$, then a.s. $(S, S+u_1), \cdots, (S, S+u_n)$ are in $\mathscr{D}^+ \cap \mathscr{D}^-$. Take a family $\{f_1, g_1, \cdots, f_n, g_n\}$ of bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}$ and let $A \in \mathcal{G}_S$. By (12), we have

$$E\Big[\prod_{i=1}^n f_i(\varphi^+_{S,S+u_i}(0)) g_i(\varphi^-_{S,S+u_i}(0)) 1_A\Big] = \lim_{q\to\infty} E\Big[\prod_{i=1}^n f_i(\varphi^+_{S_q,S_{q,u_i}}(0)) g_i(\varphi^-_{S_q,S_{q,u_i}}(0)) 1_A\Big].$$

For q large enough ($\frac{2}{q} < u_1$), we have

$$E\Big[\prod_{i=1}^n f_i(\varphi^+_{S_q,S_{q,u_i}}(0)) g_i(\varphi^-_{S_q,S_{q,u_i}}(0)) 1_A\Big]$$

$$= \sum_{m\geq 0} E\Big[\prod_{i=1}^n f_i\left(\varphi^+_{\frac{m+1}{q}, \frac{m-1}{q}+u_i}(0)\right) g_i\left(\varphi^-_{\frac{m+1}{q}, \frac{m-1}{q}+u_i}(0)\right) 1_{A\cap\{\frac{m}{q}\leq S<\frac{m+1}{q}\}}\Big]$$

with $A\cap\{\frac{m}{q} \leq S < \frac{m+1}{q}\} \in \mathcal{G}_{\frac{m+1}{q}} \subset \sigma(\varphi^+_{u,v}(z), \varphi^-_{u,v}(z), z \in \mathscr{C}, 0 \leq u \leq v \leq \frac{m+1}{q})$. Now using the independence of increments and the stationarity of (φ^+, φ^-), the second step easily holds.

Third step. φ solves $(T_{\mathscr{C}})$.

Denote ρ_0^k simply by ρ^k. For all $k \in \mathbb{N}$, a.s. $u \longmapsto \varphi_{\rho^k,u}(z)$ is continuous on $[\rho^k, \rho^{k+1}]$ for all $z \in \mathscr{C}$. Consequently for all $z \in \mathscr{C}$, a.s. $u \longmapsto \varphi_{0,u}(z)$ is continuous on $[0, +\infty[$ and in particular, $\varphi_{0,\rho^k}(z)$ is $\mathcal{G}_{\rho^k}$ measurable. Now fix $f \in C^2(\mathscr{C}), t \geq 0, z \in \mathscr{C}$ and define for all $y \in \mathscr{C}$,

$$\begin{aligned} H_{(f,t)}(y) &= f(\varphi_{\rho^1,\rho^1+t\wedge(\rho^2-\rho^1)}(y)) - f(y) - \int_0^{t\wedge(\rho^2-\rho^1)} (f'\epsilon)(\varphi_{\rho^1,\rho^1+u}(y)) dW_{\rho^1,\rho^1+u} \\ &- \frac{1}{2}\int_0^{t\wedge(\rho^2-\rho^1)} f''(\varphi_{\rho^1,\rho^1+u}(y)) du. \end{aligned}$$

Then a.s. $y \longmapsto H_{(f,t)}(y)$ is measurable from $\mathscr{C}$ into $\mathbb{R}$. Moreover $H_{(f,t)}$ is $\sigma(\varphi_{\rho^1,(\rho^1+u)\wedge\rho^2}, u \geq 0)$ measurable and $H_{(f,t)}(y) = 0$ a.s. for all $y \in \mathscr{C}$ by the first step. The second step yields $H_{(f,t)}(\varphi_{0,\rho^1}(z)) = 0$ a.s. and we may replace y by $\varphi_{0,\rho^1}(z)$ directly in the stochastic integral so that, using the flow property, we get

$$\begin{aligned} f\left(\varphi_{0,\rho^1+t\wedge(\rho^2-\rho^1)}(z)\right) &= f(\varphi_{0,\rho^1}(z)) + \int_0^{t\wedge(\rho^2-\rho^1)} (f'\epsilon)(\varphi_{0,\rho^1+u}(z)) dW_{\rho^1,\rho^1+u} \\ &+ \frac{1}{2}\int_0^{t\wedge(\rho^2-\rho^1)} f''(\varphi_{0,\rho^1+u}(z)) du \\ &= f(z) + \int_0^{\rho^1+t\wedge(\rho^2-\rho^1)} (f'\epsilon)(\varphi_{0,u}(z)) dW_u + \frac{1}{2}\int_0^{\rho^1+t\wedge(\rho^2-\rho^1)} f''(\varphi_{0,u}(z)) du. \end{aligned}$$

By induction, we have a.s. $\forall k \in \mathbb{N}$,

$$\begin{aligned} f(\varphi_{0,\rho^k+t\wedge(\rho^{k+1}-\rho^k)}(z)) &= f(z) + \int_0^{\rho^k+t\wedge(\rho^{k+1}-\rho^k)} (f'\epsilon)(\varphi_{0,u}(z)) dW_u \\ &+ \frac{1}{2}\int_0^{\rho^k+t\wedge(\rho^{k+1}-\rho^k)} f''(\varphi_{0,u}(z)) du. \end{aligned}$$

This implies that φ solves $(T_{\mathscr{C}})$. The fact that K^{m^+,m^-} solves $(T_{\mathscr{C}})$ is similar to Proposition 4.1 (ii) in [7] using Proposition 4. $\square$

3 Flows solution of $(T_{\mathscr{C}})$

From now on (K, W) is a solution of $(T_{\mathscr{C}})$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Fix $s \in \mathbb{R}, z \in \mathscr{C}$, then $(K_{s,t}(z))_{t\geq s}$ can be modified such that, a.s. the mapping $t \longmapsto K_{s,t}(z)$ is continuous from $[s, +\infty[$ into $\mathcal{P}(\mathscr{C})$. We will always consider this modification for $(K_{s,t}(z))_{t\geq s}$ for all fixed s and z.

Lemma 3. *(i) For all $z \in \mathscr{C}$ and $s \in \mathbb{R}$, denote $\tau_s(z) = \inf\{r \geq s,\ ze^{i\epsilon(z)W_{s,r}} = 1$ or $e^{il}\}$. Then a.s.*

$$K_{s,t}(z) = \delta_{ze^{i\epsilon(z)W_{s,t}}},\ \textit{if}\ \ s \leq t \leq \tau_s(z).$$

(ii) $\sigma(W) \subset \sigma(K)$.

Proof. (i) We follow Lemma 3.1 [7]. Define

$$\mathscr{C}^+ = \{z \in \mathscr{C} : \arg(z) \in]0, l[\},\ \mathscr{C}^- = \mathscr{C} \setminus \mathscr{C}^+. \tag{13}$$

For $z \in \mathscr{C}^+$, let

$$\tilde{\tau}_z = \inf\left\{t \geq 0 : K_{0,t}(z, \mathscr{C}^-) > 0\right\}.$$

Let $f \in C^2(\mathscr{C})$ such that $f(y) = \arg(y)$ if $y \in \mathscr{C}^+$ By applying f in $(T_{\mathscr{C}})$, we have for $t < \tilde{\tau}_z$,

$$K_{0,t}f(z) = f(z) + W_t$$

and thus, for $t < \tilde{\tau}_z$,

$$\int_{\mathscr{C}} \arg(y) K_{0,t}(z, dy) = \arg(z) + W_t. \tag{14}$$

By applying f^2 in $(T_{\mathscr{C}})$, we also have for $t < \tilde{\tau}_z$,

$$K_{0,t}f^2(z) = f^2(z) + 2\int_0^t \int_{\mathscr{C}} \arg(y) K_{0,u}(z, dy) dW_u + t.$$

Using (14), we obtain that for $t < \tilde{\tau}_z$,

$$\int_{\mathscr{C}} (\arg(y) - \arg(z) - W_t)^2 K_{0,t}(z, dy) = 0.$$

By continuity a.s.

$$K_{0,t}(z) = \delta_{ze^{i\epsilon(z)W_t}} \text{ for all } t \in [0, \tilde{\tau}_z].$$

The fact that $\tau_0(z) = \tilde{\tau}_z$ easily follows.

(ii) Let $(f_n)_{n\geq 1}$ be a sequence in $C^2(\mathscr{C})$ such that $f_n'(z) \to \epsilon(z)$ as $n \to \infty$ for all $z \in \mathscr{C} \setminus \{1, e^{il}\}$. Applying f_n in $(T_{\mathscr{C}})$, we get

$$\int_0^t K_{0,u}(\epsilon f_n')(1)dW_u = K_{0,t}f_n(1) - f_n(1) - \frac{1}{2}\int_0^t K_{0,u}f_n''(1)du.$$

It is easy to check that $\int_0^t K_{0,u}(\epsilon f_n')(1)dW_u$ converges towards W_t in $L^2(\mathbb{P})$ as $n \to \infty$ whence

$$W_t = \lim_{n\to\infty}\left(K_{0,t}f_n(1) - f_n(1) - \frac{1}{2}\int_0^t K_{0,u}f_n''(1)du\right) \text{ in } L^2(\mathbb{P})$$

which proves (ii). $\square$

3.1 Unicity of the Wiener solution.

Our aim in this section is to prove that $(T_{\mathscr{C}})$ admits only one Wiener solution (i.e. such that $\sigma(W) \subset \sigma(K)$). This solution is K^{m^+,m^-} with $m^+ = m^- = \delta_{\frac{1}{2}}$. For this, we will essentially follow the general idea of [5]: the Wiener solution is unique because its Wiener chaos decomposition can be given (see (15) and (16) below). Let p be semigroup of the standard Brownian motion on $\mathbb{R}$. Then the semigroup of the Brownian motion on $\mathscr{C}$ writes

$$P_t(e^{ix}, e^{iy}) = \sum_{k\in\mathbb{Z}} p_t(x, y + 2k\pi), \quad x, y \in [0, 2\pi[.$$

For all $f \in C^1(\mathscr{C})$, we easily check that $P_tf \in C^1(\mathscr{C})$ and $(P_tf)' = P_tf'$. Let $Af = \frac{1}{2}f'', f \in C^2(\mathscr{C})$ be the generator of P.

Proposition 7. *Equation $(T_{\mathscr{C}})$ has at most one Wiener solution: If (K, W) is a solution such that $\sigma(W) \subset \sigma(K)$, then $\forall t \geq 0, f \in C^\infty(\mathscr{C})$ and all $z \in \mathscr{C}$,*

$$K_{0,t}f(z) = P_tf(z) + \sum_{n=1}^{\infty} J_t^n f(z) \text{ in } L^2(\mathbb{P}) \tag{15}$$

where

$$J_t^n f(z) = \int_{0<s_1<\cdots<s_n<t} P_{s_1}(D(P_{s_2-s_1}\cdots D(P_{t-s_n}f)))(z)dW_{0,s_1}\cdots dW_{0,s_n} \tag{16}$$

no longer depends on K and $Df(z) = \epsilon(z)f'(z)$.

Proof. Let (K, W) be a solution of $(T_{\mathscr{C}})$ (not necessarily a Wiener flow). Our first aim is to establish the following

Lemma 4. *Fix $f \in C^\infty(\mathscr{C})$ and $z \in \mathscr{C}$. Then*

$$K_{0,t}f(z) = P_t f(z) + \int_0^t K_{0,u}(D(P_{t-u}f))(z)dW_u.$$

Proof. Let $f \in C^\infty(\mathscr{C}), z \in \mathscr{C}$ and denote $K_{0,t}$ simply by K_t. Note that $\int_0^t K_u(D(P_{t-u}f))(z)dW_u$ is well defined:

$$\int_0^t E[K_u(D(P_{t-u}f))(z)]^2 du \leq \int_0^t P_u((D(P_{t-u}f))^2)(z)du \leq \int_0^t ||(P_{t-u}f)'||_\infty^2 du$$

and the right-hand side is smaller than $t||f'||_\infty^2$. Now

$$K_t f(z) - P_t f(z) - \int_0^t K_u(D(P_{t-u}f))(z)dW_u = \sum_{p=0}^{n-1}(K_{\frac{(p+1)t}{n}} P_{t-\frac{(p+1)t}{n}} f - K_{\frac{pt}{n}} P_{t-\frac{pt}{n}} f)(z)$$

$$-\sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})f)(z)dW_u - \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D(P_{t-\frac{(p+1)t}{n}} f)(z)dW_u.$$

For all $p \in \{0, .., n-1\}$, set $f_{p,n} = P_{t-\frac{(p+1)t}{n}} f \in C^\infty(\mathscr{C})$ and so by replacing f by $f_{p,n}$ in $(T_\mathscr{C})$, we get

$$\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u(Df_{p,n})(z)dW_u = K_{\frac{(p+1)t}{n}} f_{p,n}(z) - K_{\frac{pt}{n}} f_{p,n}(z) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u(Af_{p,n})(z)du$$

$$= K_{\frac{(p+1)t}{n}} f_{p,n}(z) - K_{\frac{pt}{n}} f_{p,n}(z) - \frac{t}{n} K_{\frac{pt}{n}}(Af_{p,n})(z) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_u - K_{\frac{pt}{n}})(Af_{p,n})(z)du.$$

Then we can write

$$K_t f(z) - P_t f(z) - \int_0^t K_u(D(P_{t-u}f))(z)dW_u = A_1(n) + A_2(n) + A_3(n),$$

where

$$\begin{aligned}
A_1(n) &= -\sum_{p=0}^{n-1} K_{\frac{pt}{n}}[P_{t-\frac{pt}{n}} f - P_{t-\frac{(p+1)t}{n}} f - \frac{t}{n} A P_{t-\frac{(p+1)t}{n}} f](z), \\
A_2(n) &= -\sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})f)(z)dW_u, \\
A_3(n) &= \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_u - K_{\frac{pt}{n}}) A P_{t-\frac{(p+1)t}{n}} f(z)du.
\end{aligned}$$

Using $||K_u g||_\infty \leq ||g||_\infty$ for g a bounded measurable function, we obtain

$$|A_1(n)| \leq \sum_{p=0}^{n-1} \left|\left| P_{t-\frac{(p+1)t}{n}}[P_{\frac{t}{n}} f - f - \frac{t}{n} Af] \right|\right|_\infty \leq n \left|\left| P_{\frac{t}{n}} f - f - \frac{t}{n} Af \right|\right|_\infty = t \left|\left| \frac{P_{\frac{t}{n}} f - f}{\frac{t}{n}} - Af \right|\right|_\infty.$$

Since $f \in C^\infty(\mathscr{C})$, this shows that $A_1(n)$ converges to 0 as $n \to \infty$. Note that $A_2(n)$ is the sum of orthogonal terms in $L^2(\mathbb{P})$. Consequently

$$||A_2(n)||^2_{L^2(\mathbb{P})} = \sum_{p=0}^{n-1} \left|\left| \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})f)(z) dW_u \right|\right|^2_{L^2(\mathbb{P})}.$$

By applying Jensen's inequality, we arrive at

$$||A_2(n)||^2_{L^2(\mathbb{P})} \leq \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_u V_u^2(z) du$$

where $V_u = (P_{t-u}f)' - (P_{t-\frac{(p+1)t}{n}}f)' = P_{t-u}f' - P_{t-\frac{(p+1)t}{n}}f'$. For all $u \in [\frac{pt}{n}, \frac{(p+1)t}{n}]$, we have

$$P_u V_u^2(z) \leq ||V_u||^2_\infty = \left|\left| P_{t-\frac{(p+1)t}{n}} \left(P_{\frac{(p+1)t}{n}-u} f' - f' \right) \right|\right|^2_\infty \leq ||P_{\frac{(p+1)t}{n}-u} f' - f'||^2_\infty.$$

Consequently

$$||A_2(n)||^2_{L^2(\mathbb{P})} \leq \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{\frac{(p+1)t}{n}-u} f' - f'||^2_\infty du = n \int_0^{\frac{t}{n}} ||P_u f' - f'||^2_\infty du,$$

and one can deduce that $A_2(n)$ tends to 0 as $n \to +\infty$ in $L^2(\mathbb{P})$. Now

$$||A_3(n)||_{L^2(\mathbb{P})} \leq \sum_{p=0}^{n-1} \left|\left| \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_u - K_{\frac{pt}{n}}) A P_{t-\frac{(p+1)t}{n}} f(z) du \right|\right|_{L^2(\mathbb{P})}.$$

Set $h_{p,n} = AP_{t-\frac{(p+1)t}{n}}f$. Then $h_{p,n} \in C^\infty(\mathscr{C})$ for all $p \in [0, n-1]$. By the Cauchy-Schwarz inequality

$$||A_3(n)||_{L^2(\mathbb{P})} \leq \sqrt{t} \left\{ \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} E[((K_u - K_{\frac{pt}{n}}) h_{p,n}(z))^2] du \right\}^{\frac{1}{2}}.$$

If $u \in [\frac{pt}{n}, \frac{(p+1)t}{n}]$:

$$\begin{aligned}
E[((K_u - K_{\frac{pt}{n}}) h_{p,n}(z))^2] &\leq E[K_{\frac{pt}{n}} (K_{\frac{pt}{n},u} h_{p,n} - h_{p,n})^2(z)] \\
&\leq E[K_{\frac{pt}{n}} (K_{\frac{pt}{n},u} h^2_{p,n} - 2h_{p,n} K_{\frac{pt}{n},u} h_{p,n} + h^2_{p,n})(z)] \\
&\leq P_{\frac{pt}{n}} \left(P_{u-\frac{pt}{n}} h^2_{p,n} - 2h_{p,n} P_{u-\frac{pt}{n}} h_{p,n} + h^2_{p,n} \right)(z) \\
&\leq ||P_{u-\frac{pt}{n}} h^2_{p,n} - 2h_{p,n} P_{u-\frac{pt}{n}} h_{p,n} + h^2_{p,n}||_\infty \\
&\leq 2||h_{p,n}||_\infty ||P_{u-\frac{pt}{n}} h_{p,n} - h_{p,n}||_\infty + ||P_{u-\frac{pt}{n}} h^2_{p,n} - h^2_{p,n}||_\infty.
\end{aligned}$$

Therefore $||A_3(n)||_{L^2(\mathbb{P})} \leq \sqrt{t}(2C_1(n) + C_2(n))^{\frac{1}{2}}$, where

$$C_1(n) = \sum_{p=0}^{n-1} ||h_{p,n}||_\infty \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}} h_{p,n} - h_{p,n}||_\infty du$$

and

$$C_2(n) = \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}} h_{p,n}^2 - h_{p,n}^2||_\infty du.$$

From $||h_{p,n}||_\infty \leq ||Af||_\infty$ and $||P_{u-\frac{pt}{n}} h_{p,n} - h_{p,n}||_\infty \leq ||P_{u-\frac{pt}{n}} Af - Af||_\infty$, we get

$$C_1(n) \leq ||Af||_\infty \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}} Af - Af||_\infty du \leq ||Af||_\infty \int_0^t ||P_{\frac{s}{n}} Af - Af||_\infty ds.$$

As $Af \in C^\infty(\mathscr{C})$, $C_1(n)$ tends to 0 obviously. On the other hand, $h_{p,n}^2 \in C^\infty(\mathscr{C})$ and so

$$C_2(n) = \frac{1}{n} \sum_{p=0}^{n-1} \int_0^t ||P_{\frac{s}{n}} h_{p,n}^2 - h_{p,n}^2||_\infty ds \leq \frac{1}{n} \sum_{p=0}^{n-1} \int_0^t \left(\int_0^{\frac{s}{n}} ||Ah_{p,n}^2||_\infty du \right) ds.$$

Now we easily verify that $h_{p,n}, h'_{p,n}, h''_{p,n}$ are uniformly bounded with respect to n and $0 \leq p \leq n-1$. As a result $C_2(n)$ tends to 0 as $n \to \infty$. This establishes Lemma 4. $\square$

Assume that (K, W) is a Wiener solution of $(T_\mathscr{C})$ and for $t \geq 0, f \in C^\infty(\mathscr{C})$ and $z \in \mathscr{C}$, let $K_{0,t}f(z) = P_t f(z) + \sum_{n=1}^\infty J_t^n f(z)$ be the decomposition in Wiener chaos of $K_{0,t}f(z)$ in L^2 sense. By iterating the identity of Lemma 4, we see that for all $n \geq 1$, $J_t^n f(z)$ is given by (16). $\square$

Consequence: Let K^W be the unique Wiener solution of $(T_\mathscr{C})$. Since $\sigma(W) \subset \sigma(K)$, we can define K^* the stochastic flow obtained by filtering K with respect to $\sigma(W)$ (Lemma 3-2 (ii) in [6]). Then, for all $s \leq t$ and all $z \in \mathscr{C}$, a.s.

$$K_{s,t}^*(z) = E[K_{s,t}(z)|\sigma(W)].$$

As a result, (K^*, W) solves also $(T_\mathscr{C})$ and by the last proposition, for all $s \leq t$ and all $z \in \mathscr{C}$, a.s.

$$E[K_{s,t}(z)|\sigma(W)] = K_{s,t}^W(z). \tag{17}$$

3.2 Proof of Theorem 1 (2)

Using the flow property and the independence of the increments satisfied by K, it is easily seen that the law of $(K_{0,t_1}, \cdots, K_{0,t_n})$ for all $(t_1, \cdots, t_n) \in (\mathbb{R}_+)^n$ and therefore the law of K is uniquely determined by the knowledge of the law of $K_{0,t}$ for all $t \geq 0$. In the sequel, we will show the existence of two probability measures m^+ and m^- on $[0, 1]$ with mean $\frac{1}{2}$ such that for all $t \geq 0, K_{0,t}^{m^+,m^-} \overset{law}{=} K_{0,t}$ which will imply Part (2) of Theorem 1.

3.2.1 A stochastic flow of mappings associated to K.

Let $P_t^n = E[K_{0,t}^{\otimes n}]$ be the compatible family of Feller semigroups associated to K and let $(P^{n,c})_{n\geq 1}$ be the family of compatible Markov semigroups associated to $(P^n)_{n\geq 1}$ by Theorem 41 [6]. Then we have

Lemma 5. *$(P^{n,c})_{n\geq 1}$ is a compatible family of Feller semigroups associated with a flow of mappings φ^c.*

Proof. For each $(x,y) \in \mathscr{C}^2$, let $(X_t^x, Y_t^y)_{t\geq 0}$ be the two point motion started at (x,y) associated with P^2 constructed as in Section 2.6 [6] on an extension $(\Omega\times\Omega', \mathcal{E}, \mathbb{Q})$ of $(\Omega, \mathcal{A}, \mathbb{P})$ such that the law of (X_t^x, Y_t^y) given $\omega \in \Omega$ is $K_{0,t}(x) \otimes K_{0,t}(y)$. Define

$$T^{x,y} := \inf\{t \geq 0 : X_t^x = Y_t^y\}.$$

By Theorem 4.1 [6], we only need to check that: for all $t > 0, \varepsilon > 0$ and $x \in \mathscr{C}$,

$$\lim_{y\to x} \mathbb{Q}(\{T^{x,y} > t\} \cap \{d(X_t^x, Y_t^y) > \varepsilon\}) = 0 \qquad (C).$$

Fix $t > 0$ and $\epsilon > 0$.

First case $x = 1$. Recall that for all $0 \leq t \leq \rho$ $(:= \rho_0)$,

$$K_{0,t}^W(1) = \frac{1}{2}(\delta_{e^{iW_t^+}} + \delta_{e^{-iW_t^+}}).$$

This shows that $K_{0,t}(1)$ is supported on $\{e^{iW_t^+}, e^{-iW_t^+}\}$ and so $X_t^1 = e^{iW_t^+}$ or $e^{-iW_t^+}$ when $t \leq \rho$. Moreover, if $y \notin \{1, e^{il}\}$, then $X_t^y = ye^{i\varepsilon(y)W_t}$ when $0 \leq t \leq \tau(y)(:= \tau_0(y))$ by Lemma 3 (i).
Let $A = \{T^{1,y} > t\} \cap \{d(X_t^1, Y_t^y) > \varepsilon\}$ where y is close to 1 and $y \neq 1$. Write

$$\mathbb{Q}(A) = \mathbb{Q}(A \cap \{t \leq \tau(y)\}) + \mathbb{Q}(A \cap \{t > \tau(y)\}).$$

Since $\tau(y)$ tends to 0 as y tends to 1, we have $\lim_{y\to 1} \mathbb{Q}(A \cap \{t \leq \tau(y)\}) = 0$. Moreover

$$\mathbb{Q}(A \cap \{t > \tau(y)\}) \leq \mathbb{Q}(B) + \mathbb{Q}(X_{\tau(y)}^y = e^{il}).$$

where $B = A \cap \{t > \tau(y), X_{\tau(y)}^y = 1\}$. Obviously

$$\mathbb{Q}(B) \leq \mathbb{Q}(B \cap \{\tau(y) < \rho\}) + \mathbb{Q}(\tau(y) \geq \rho)$$

with $\lim_{y\to 1} \mathbb{Q}(\tau(y) \geq \rho) = 0$. On $B \cap \{\tau(y) < \rho\}$, we have $X_{\tau(y)}^1 = X_{\tau(y)}^y = 1$ and thus $T^{1,y} \leq \tau(y)$. As a result

$$\mathbb{Q}(B \cap \{\tau(y) < \rho\}) \leq \mathbb{Q}(t < T^{1,y} \leq \tau(y)).$$

Since the right-hand side converges to 0 as $y \to 1$, (C) is satisfied for $x = 1$.

Second case $x \neq 1$. By analogy (C) is satisfied for $x = e^{il}$. Let $x \notin \{1, e^{il}\}$ and y be close to x, then X^x and X^y move parallely until one of the two processes reaches 1 or e^{il} say at time T. Since P^2 is Feller, the strong Markov property at time T and the established result for $x \in \{1, e^{il}\}$ allows to deduce (C) for x. $\square$

Consequence: By the proof of Theorem 4.2 [6], there exists a joint realization (K^1, K^2) on a probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ where K^1 and K^2 are two stochastic flows of kernels satisfying $K^1 \overset{law}{=} \delta_{\varphi^c}$, $K^2 \overset{law}{=} K$ and such that:

(i) $\hat{K}_{s,t}(x,y) = K^1_{s,t}(x) \otimes K^2_{s,t}(y)$ is a stochastic flow of kernels on $\mathscr{C}^2$,

(ii) For all $s \leq t, z \in \mathscr{C}$, a.s. $K^2_{s,t}(z) = E[K^1_{s,t}(z)|K^2]$.

To simplify notations, we will denote (K^1, K^2) by (δ_{φ^c}, K). Recall that (i) and (ii) are also satisfied by the pair $(\delta_\varphi, K^{m^+,m^-})$ constructed in Section 2.3. Now (ii) rewrites, for all $s \leq t, z \in \mathscr{C}$,

$$K_{s,t}(z) = E[\delta_{\varphi^c_{s,t}(z)}|K] \quad a.s. \tag{18}$$

and using (17), we obtain, for all $s \leq t, z \in \mathscr{C}$,

$$K^W_{s,t}(z) = E[\delta_{\varphi^c_{s,t}(z)}|\sigma(W)] \ a.s. \tag{19}$$

with K^W being the Wiener solution.

3.2.2 The law of K.

Recall the definitions of $\mathscr{C}^+$ and $\mathscr{C}^-$ from (13) and set for all $s \leq t$,

$$U^+_{s,t} = K_{s,t}(1, \mathscr{C}^+) \text{ and } U^-_{s,t} = K_{s,t}(e^{il}, \mathscr{C}^-).$$

Proposition 8. *Recall the definition of ρ_s from (9).*

(i) There exist two probability measures m^+ and m^- on $[0,1]$ with mean $\frac{1}{2}$ such that for all $s < t$, conditionally to $\{s < t < \rho_s\}$, $U^\pm_{s,t}$ is independent of W and has for law $m^\pm$. Moreover, for all $s \in \mathbb{R}, z \in \mathscr{C}$, a.s. $\forall t \in [s, \rho_s]$,

$$\begin{aligned} K_{s,t}(z) &= \delta_{ze^{i\epsilon(z)W_{s,t}}} 1_{\{t \leq \tau_s(z)\}} \\ &+ \left(K_{s,t}(1) 1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}}=1\}} + K_{s,t}(e^{il}) 1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}}=e^{il}\}}\right) 1_{\{t > \tau_s(z)\}} \end{aligned}$$

where

$$\begin{aligned} K_{s,t}(1) &= U^+_{s,t}\delta_{\exp(iW^+_{s,t})} + (1-U^+_{s,t})\delta_{\exp(-iW^+_{s,t})}, \\ K_{s,t}(e^{il}) &= U^-_{s,t}\delta_{\exp(i(l+W^-_{s,t}))} + (1-U^-_{s,t})\delta_{\exp(i(l-W^-_{s,t}))}. \end{aligned}$$

(ii) For all $s<t$, conditionally to $\{\rho_s > t\}$, $U^+_{s,t}, U^-_{s,t}$ and W are independent.

The proof of (i) essentially follows [7] and will be deduced after establishing the lemmas 6,7,8,9 and 10 below.

For all $-\infty \leq s \leq t \leq +\infty$, define $\mathcal{F}^K_{s,t} = \sigma(K_{u,v}, s \leq u \leq v \leq t)$ and recall the definition of $\mathcal{F}^W_{s,t}$ from (6). When $s=0$, we denote $K_{0,t}, \varphi^c_{0,t}, \mathcal{F}^K_{0,t}, \mathcal{F}^W_{0,t}, U^{\pm}_{0,t}$ simply by $K_t, \varphi^c_t, \mathcal{F}^K_t, \mathcal{F}^W_t, U^{\pm}_t$ We will always consider the usual augmentations of these σ-fields which include all $\mathbb{P}$-negligible sets and are right-continuous. For each each $z \in \mathscr{C}$, recall that $t \longmapsto K_t(z)$ is continuous from $[0,+\infty[$ into $\mathcal{P}(\mathscr{C})$. Denote by $\mathbb{P}_z$ the law of $K_\cdot(z)$ which is a probability measure on $C(\mathbb{R}_+, \mathcal{P}(\mathscr{C}))$, then since $K_\cdot(z)$ is a Feller process (see Lemma 2.2 [6]) the following strong Markov property holds

Lemma 6. *Let $z_1, z_2 \in \mathscr{C}$ and T be a finite $(\mathcal{F}^K_t)_{t\geq 0}$-stopping time. On $\{K_T(z_1) = \delta_{z_2}\}$, the law of $K_{T+\cdot}(z_1)$ knowing $\mathcal{F}^K_T$ is given by $\mathbb{P}_{z_2}$.*

Let

$$\rho^+ = \inf\{r \geq 0 : W^+_r = l\} \text{ and } L = \sup\{r \in [0,\rho^+] : W^+_r = 0\}.$$

Thanks to (19), on the event $\{0 \leq t \leq \rho^+\}$, a.s.

$$E[\delta_{\varphi^c_t(1)} | \sigma(W)] = \frac{1}{2}(e^{iW^+_t} + e^{-iW^+_t}).$$

By the continuity of $\varphi^c_\cdot(1)$, this shows that a.s.

$$\forall t \in [0,\rho^+],\ \varphi^c_t(1) \in \{e^{iW^+_t}, e^{-iW^+_t}\}. \tag{20}$$

Let $h \in C(\mathscr{C})$ such that $\forall x \in [-l,l]$, $h(e^{ix}) = |x|$. Using (18), the fact that $\sigma(W) \subset \sigma(K)$ and the continuity of $t \longmapsto K_t(1)$, we have a.s. $\forall g \in C_0(\mathbb{R}), \forall t \in [0,\rho^+]$,

$$K_t(g \circ h)(1) = g(W^+_t).$$

Thus a.s. $\forall t \in [0,\rho^+]$, $K_t h(1) = W^+_t$ and ρ^+ can be expressed as

$$\rho^+ = \inf\{t \geq 0 : K_t h(1) = l\}. \tag{21}$$

Define the σ-fields:

$$\mathcal{F}_{L-} = \sigma(X_L, X \text{ is a bounded } \mathcal{F}^W_{0,\cdot} - \text{previsible process}),$$

$$\mathcal{F}_{L+} = \sigma(X_L, X \text{ is a bounded } \mathcal{F}^W_{0,\cdot} - \text{progressive process}).$$

By Lemma 4.11 in [7], we have $\mathcal{F}_{L+} = \mathcal{F}_{L-}$ (see also [3]). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function and set

$$X_t = E[f(U_t^+)|\sigma(W)]1_{\{0 \le t \le \rho^+\}}.$$

By (18), the process U^+ is constant on the excursions of W^+ out of 0 before ρ^+.

Lemma 7. *There exists an $\mathcal{F}^W$-progressive version of X denoted Y that is constant on the excursions of W^+ out of 0 before ρ^+ and satisfies $Y_L = Y_{\rho^+}$ a.s.*

Proof. We closely follow Lemma 4.12 [7] and correct an error at the end of the proof there. By induction, for all integers k and n, define the sequence of stopping times $S_{k,n}$ and $T_{k,n}$ by the relations: $T_{0,n} = 0$ and for $k \ge 1$,

$$\begin{aligned} S_{k,n} &= \inf\{t \ge T_{k-1,n} : W_t^+ = 2^{-n}\}, \\ T_{k,n} &= \inf\{t \ge S_{k,n} : W_t^+ = 0\}. \end{aligned}$$

In the following $U^+_{k,n}$ will denote $U^+_{S_{k,n}}$. For all $t > 0$, on $\{t \in [S_{k,n}, T_{k,n}[, t \le \rho^+\}$, we have $U_t^+ = U^+_{k,n}$ as. Let $X_{k,n} := E[f(U^+_{k,n})|W]1_{\{S_{k,n} \le \rho^+\}}$. Since $\sigma(W_{S_{k,n}, u+S_{k,n}}, u \ge 0)$ is independent of $\mathcal{F}^K_{S_{k,n}}$, we have $X_{k,n} = E[f(U_{k,n})|\mathcal{F}^W_{S_{k,n}}]1_{\{S_{k,n} \le \rho^+\}}$ which is $\mathcal{F}^W_{S_{k,n}}$ measurable. Set $I_n = \bigcup_{k \ge 1}[S_{k,n}, T_{k,n}[$ and define

$$X_t^n = \begin{cases} X_{k,n} & \text{if } t \in [S_{k,n}, T_{k,n}[\text{ (for some } k) \text{ and } t \le \rho^+, \\ f(0) & \text{if } t \in I_n^c \cap [0, \rho^+], \\ 0 & \text{if } t > \rho^+. \end{cases}$$

Then X^n is $\mathcal{F}^W$-progressive. For all $t \ge 0$, set $\tilde{X}_t = \limsup_{n \to \infty} X_t^n$. The process $\tilde{X}$ is $\mathcal{F}^W$-progressive and for all $t \ge 0$, $\tilde{X}_t = X_t$ a.s. Indeed, fix $t > 0$, then on $\{\rho^+ > t\}$, choose k_0 and n_0 such that $t \in [S_{k_0,n_0}, T_{k_0,n_0}[$, then $X_t^{n_0} = X_{k_0,n_0}$. For all $n \ge n_0$, there exists an integer l_n such that $t \in [S_{l_n,n}, T_{l_n,n}[$. Thus $X_t^n = X_{l_n,n} = X_{k_0,n_0}$ since S_{k_0,n_0} and $S_{l_n,n}$ belong to the same excursion interval of W^+ containing also t. Now set $Y_0 = f(0)$ and $Y_t = \limsup_{n \to \infty} \tilde{X}_{t+\frac{1}{n}}$ for all $t > 0$. Then Y is a modification of X which is $\mathcal{F}^W$-progressive and constant on the excursions of W^+ out of 0 before ρ^+. Moreover $Y_L = Y_{\rho^+}$ a.s. $\square$

We take for X this $\mathcal{F}^W$-progressive version. Then $X_{\rho^+} = E[f(U^+_{\rho^+})|\sigma(W)]$ is $\mathcal{F}_{L+}$ measurable.

Lemma 8. $E[X_{\rho^+}|\mathcal{F}_{L-}] = E[f(U^+_{\rho^+})]$.

Proof. Let S be an $\mathcal{F}^W$-stopping time and $d_S = \inf\{t \geq S : W^+_t = 0\}$. We have $\{S < L\} = \{d_S < \rho^+\}$ (up to some negligible set) and so $\{S < L\} \in \mathcal{F}^W_{d_S}$. Let $H = d_S \wedge \rho^+$ and $K = \inf\{r \geq 0 : K_{H+r}h(1) = l\}$, then

$$E[X_{\rho^+} 1_{\{d_S<\rho^+\}}] = E[f(U^+_{H+K}) 1_{\{d_S<\rho^+, K_H(1)=\delta_1\}}].$$

Note that on $\{d_S < \rho^+\}$, we have $H + K = \rho^+$ a.s. Applying Lemma 6 at time H and using (21), we get

$$E[X_{\rho^+} 1_{\{d_s<\rho^+\}}] = E[f(U^+_{\rho^+})] E[1_{\{d_S<\rho^+, K_H(1)=\delta_1\}}] = E[f(U^+_{\rho^+})] \mathbb{P}(d_S < \rho^+).$$

Since the σ-field $\mathcal{F}_{L-}$ is generated by the events $\{S < L\}$ for all stopping time S (see [8] page 344), the lemma holds. $\square$

The previous lemma implies that $U^+_{\rho^+}$ is independent of $\sigma(W)$ (Lemma 4.14 [7]) and the same holds if we replace ρ^+ by $\inf\{t \geq 0 : W^+_t = a\}$ where $0 < a \leq l$. For n such that $2^{-n} < l$, define inductively $T^+_{0,n} = 0$ and for $k \geq 1$:

$$\begin{aligned} S^+_{k,n} &= \inf\{t \geq T^+_{k-1,n} : W^+_t = 2^{-n}\}, \\ T^+_{k,n} &= \inf\{t \geq S^+_{k,n} : W^+_t = 0\}. \end{aligned}$$

Set $V^+_{k,n} = U^+_{S^+_{k,n}}$. Then, we have the following

Lemma 9. *For all $q \geq 1$, conditionally to $\{S^+_{q,n} \leq \rho^+\}$, $V^+_{1,n}, \cdots, V^+_{q,n}, W$ are independent and $V^+_{1,n}, \cdots, V^+_{q,n}$ have the same law (which depends on n but no longer depends on q).*

Proof. We prove the result by induction on q. For $q = 1$, this has been justified. Suppose the result holds for $q - 1$ and let (f_j) be an approximation of ϵ as in the proof of Lemma 3 (ii). For a fixed $t \geq 0$,

$$W_{T^+_{q-1,n}, t+T^+_{q-1,n}} = \lim_{j\to\infty} \left(K_{t+T^+_{q-1,n}} f_j(1) - K_{T^+_{q-1,n}} f_j(1) - \frac{1}{2}\int_0^t K_{u+T^+_{q-1,n}} f''_j(1) du \right) \text{ in } L^2(\mathbb{P}).$$

On $\{S^+_{q,n} \leq \rho^+\}$, we have $K_{T^+_{q-1,n}}(1) = \delta_1$ and therefore,

$$W_{T^+_{q-1,n}, t+T^+_{q-1,n}} = \lim_{j\to\infty} \left(K_{t+T^+_{q-1,n}} f_j(1) - f_j(1) - \frac{1}{2}\int_0^t K_{u+T^+_{q-1,n}} f''_j(1) du \right) \tag{22}$$

in $L^2(\mathbb{P}(.|S^+_{q,n} \leq \rho^+))$. As $2^{-n} < l$, $\{S^+_{q,n} \leq \rho^+\} = \{T^+_{q-1,n} \leq \rho^+\}$ a.s. Choose a family $\{g_1, \cdots, g_q, g, h\}$ of bounded continuous functions on $\mathbb{R}$. For any $A \in \mathcal{A}$, we will use the notation E_A to denote the

expectation under $\mathbb{P}(\cdot|A)$. Set $A_{q,n} = \{S^+_{q,n} \le \rho^+\}$. Using (22), an application of Lemma 6 at time $T^+_{q-1,n}$ shows that

$$E_{A_{q,n}}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})\right]$$
$$= E_{A_{q,n}}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})g(W_{t\wedge T^+_{q-1,n}})\right] E\left[h(W_t)\right] E\left[g_q(U^+_{S^+_{1,n}})\right].$$

Since $A_{q-1,n} \subset A_{q,n}$, we have by the induction hypothesis

$$E_{A_{q,n}}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})g(W_{t\wedge T^+_{q-1,n}})\right] = E_{A_{q-1,n}}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})\right] E_{A_{q,n}}\left[g(W_{t\wedge T^+_{q-1,n}})\right].$$

In conclusion

$$E_{A_{q,n}}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})\right]$$
$$= E_{A_{q-1,n}}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})\right] E_{A_{q,n}}\left[g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})\right] E\left[g_q(U^+_{S^+_{1,n}})\right].$$

The last identity remains satisfied if we replace $g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})$ by a finite product $\prod_{i=1}^k g^i(W_{t_i\wedge T^+_{q-1,n}})h^i(W_{T^+_{q-1,n},t_i+T^+_{q-1,n}})$. As a result, for all bounded continuous $g : C(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}$,

$$E_{A_{q,n}}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W)\right] = E_{A_{q-1,n}}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})\right] E_{A_{q,n}}\left[g(W)\right] E\left[g_q(U^+_{S^+_{1,n}})\right].$$

Iterating this relation, we get

$$E_{A_{q,n}}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W)\right] = \prod_{i=1}^{q} E\left[g_i(U^+_{S^+_{1,n}})\right] E_{A_{q,n}}\left[g(W)\right].$$

In particular, for all $i \in [1,q]$,

$$E_{A_{q,n}}\left[g_i(U^+_{S^+_{i,n}})\right] = E\left[g_i(U^+_{S^+_{1,n}})\right].$$

This completes the proof. □

Let m^+_n be the law of $V^+_{1,n}$ and m^+ be the law of U^+_1 under $\mathbb{P}(.|\rho^+ > 1)$. Then, we have the

Lemma 10. *The sequence $(m^+_n)_{n\ge1}$ converges weakly towards m^+. For all $t > 0$, under $\mathbb{P}(\cdot|\rho^+ > t)$, U^+_t and W are independent and the law of U^+_t is given by m^+.*

Proof. For each bounded continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\begin{aligned} E[f(U_t^+)|W]1_{\{0<t<\rho^+\}} &= \lim_{n\to\infty} \sum_k E\big[1_{\{t\in[S_{k,n}^+,T_{k,n}^+[\}} f(V_{k,n}^+)|W\big]1_{\{0<t<\rho^+\}} \\ &= \lim_{n\to\infty} \sum_k 1_{\{t\in[S_{k,n}^+,T_{k,n}^+\wedge\rho^+[\}} \left(\int f dm_n^+\right) \\ &= \left[1_{\{0<t<\rho^+\}} \lim_{n\to\infty} \int f dm_n^+\right] \end{aligned}$$

Consequently

$$\lim_{n\to\infty} \int f dm_n^+ = \frac{1}{\mathbb{P}(\rho^+ > t)} E[f(U_t^+)1_{\{\rho^+>t\}}].$$

The left-hand side no longer depends on t, which completes the proof. $\square$

We define the measure m^- by analogy. Let $\rho^- = \inf\{t \geq 0 : W_t^- = l\}$, then for all $t > 0$, under $\mathbb{P}(\cdot|\rho^- > t)$, U_t^- and W are independent and the law of U_t^- is m^-. Recall the definition $\rho_0 = \inf(\rho^+, \rho^-)$. Then, for all $t > 0$, the law of U_t^+ (respectively U_t^-) knowing $\{\rho_0 > t\}$ is given by m^+ (respectively m^-).

Now take $s = 0$ and fix $z \in \mathscr{C}$. Similarly to (20), we can deduce from (19) that a.s. for all $t \in [0, \rho_0]$,

$$\varphi_t^c(z) = ze^{i\epsilon(z)W_t}, \quad \varphi_t^c(1) \in \{e^{iW_t^+}, e^{-iW_t^+}\} \text{ and } \varphi_t^c(e^{il}) \in \{e^{i(l+W_t^-)}, e^{i(l-W_t^-)}\}.$$

Note that φ^c is constructed such that for all $x, y \in \mathscr{C}$ as. $\varphi_\cdot^c(x)$ and $\varphi_\cdot^c(y)$ collide whenever they meet. So a.s. for all $t \in [0, \rho_0]$,

$$\begin{aligned} \varphi_t^c(z) &= ze^{i\epsilon(z)W_t}1_{\{t\leq\tau_0(z)\}} \\ &+ \left(\varphi_t^c(1)1_{\{ze^{i\epsilon(z)W_{\tau_0(z)}}=1\}} + \varphi_t^c(e^{il})1_{\{ze^{i\epsilon(z)W_{\tau_0(z)}}=e^{il}\}}\right)1_{\{t>\tau_0(z)\}}, \end{aligned}$$

By (18), the second claim of Proposition 8 (i) holds.

Proof of Proposition 8 (ii) We first prove the following statements: For all $0 < s < t$, we have

(a) Conditionally to $\{s < \rho_0, t < \rho_s\}$, $U_{s,t}^+, U_{0,s}^-, W$ are independent and $U_{s,t}^+$ (resp. $U_{0,s}^-$) has for law m^+ (resp. m^-).

(b) Let

$$g_t^\pm = \sup\{u \in [0, t] : W_u^\pm = 0\}.$$

Then, conditionally to $\{g_t^- < s < g_t^+, s < \rho_0\}$, $U_{0,t}^+, U_{0,t}^-, W$ are independent and the law of $U_{0,t}^+$ (resp. $U_{0,t}^-$) is m^+ (resp. m^-).

(c) Conditionally to $\{g_t^- < g_t^+, t < \rho_0\}$, $U_{0,t}^+, U_{0,t}^-, W$ are independent.

(d) Conditionally to $\{t<\rho_0\}$, $U^+_{0,t}, U^-_{0,t}, W$ are independent.

(a) Note that $\{s<\rho_0\}\in\mathcal{F}^W_s$, $\{t<\rho_s\}\in\mathcal{F}^W_{s,+\infty}$ and $\mathcal{F}^W_{0,+\infty}=\mathcal{F}^W_s\vee\mathcal{F}^W_{s,+\infty}$ with $\mathcal{F}^W_s\subset\mathcal{F}^K_s$, $\mathcal{F}^W_{0,+\infty}\subset\mathcal{F}^K_{0,+\infty}$. Now (a) holds from Proposition 8 (i) and using the independence of $\mathcal{F}^K_s$ and $\mathcal{F}^K_{s,+\infty}$.

(b) By (a), it suffices to show that on $A=\{g^-_t<s<g^+_t, s<\rho_0\}$ (which is a subset of $\{s<\rho_0, t<\rho_s\}$), a.s. $U^-_{0,t}=U^-_{0,s}$ and $U^+_{0,t}=U^+_{s,t}$. The first equality is clear since $r\longmapsto U^-_r$ is constant on the excursions of W^- on $[0,\rho]$ and on A, s and t belong to the same excursion of W^-. Moreover, on A, we have $Z:=\varphi^c_s(1)\in\{e^{iW^+_s}, e^{-iW^+_s}\}$ and so $\mathbb{P}(\cdot|A)$ a.s.

$$\tau_s(Z)=\inf\{r\geq s: W_r-\mathrm{m}^+_{0,s}=0\}=\inf\{r\geq s: W^+_r=0\}\leq g^+_t.$$

Clearly $\varphi^c_{s,\tau_s(Z)}(Z)=\varphi^c_{s,\tau_s(Z)}(1)=1$ and therefore $\varphi^c_{s,r}(Z)=\varphi^c_{s,r}(1)$ for all $r\geq\tau_s(Z)$ (using the coalescence property of φ^c and the independence of increments). On A, $\tau_s(Z)\leq g^+_t\leq t$ and the flow property of φ^c, yields a.s.

$$\varphi^c_t(1)=\varphi^c_{s,t}(y)=\varphi^c_{s,t}(1).$$

Using (18), we get $\mathbb{P}(\cdot|A)$ a.s. $U^+_{0,t}=U^+_{s,t}$.

(c) For all $n\geq 0$, let $\mathbb{D}_n=\{\frac{k}{2^n},\ k\in\mathbb{N}\}$ and $\mathbb{D}=\cup_{n\in\mathbb{N}}\mathbb{D}_n$. Define for $0\leq u<v$,

$$n(u,v)=\inf\{n\in\mathbb{N}:\mathbb{D}_n\cap]u,v[\neq\emptyset\}\ \text{and}\ \ d(u,v)=\inf(\mathbb{D}_{n(u,v)}\cap]u,v[).$$

Then by writing

$$\{g^-_t<g^+_t, t<\rho_0\}=\bigcup_{s\in\mathbb{D}}\{g^-_t<s<g^+_t, t<\rho_0, s=d(g^-_t,g^+_t)\}$$

and using that $d(g^-_t,g^+_t)1_{\{g^-_t<g^+_t\}}$ is $\sigma(W)$-measurable, we deduce (c) from (b).

(d) By analogy with (c), conditionally to $\{g^+_t<g^-_t, t<\rho_0\}$, $U^+_{0,t}, U^-_{0,t}, W$ are independent. Now (d) holds after remarking that as. $\{t<\rho_0\}=\{g^-_t<g^+_t, t<\rho_0\}\cup\{g^+_t<g^-_t, t<\rho_0\}$

We have proved (ii) of Proposition 8 for $s=0$ which allows to deduce (ii) for all s using the stationarity of K.

Now the proof of Proposition 8 is completed.

Proposition 9. *We have* $K\overset{law}{=}K^{m^+,m^-}$.

Proof. Like in Section 2.3, extending the probability space, we can construct a flow K' such that (K', W) has the same law as (K^{m^+,m^-}, W). By Proposition 8, for all $t > s$, $K_{s,t} \stackrel{law}{=} K'_{s,t}$ conditionally to $\{\rho_s > t\}$. For $t > 0$ and $n \geq 1$, let $t_i^n = \frac{it}{n}, i \in [0, n]$ and define $A_{n,i} = \{t_i^n \leq \rho_{t_{i-1}^n}\} \in \mathcal{F}^W_{t_{i-1}^n, t_i^n}$, $A_n = \cap_{i=1}^n A_{n,i}$. Then by the independence of increments of K and K',

$$(K_{0,t_1^n}, \cdots, K_{t_{n-1}^n, t}) \stackrel{law}{=} (K'_{0,t_1^n}, \cdots, K'_{t_{n-1}^n, t}) \text{ on } A_n.$$

Recall that $\mathbb{P}(A_n^c) \to 0$ as $n \to \infty$ (see the proof of Proposition 4). Letting $n \to \infty$ and using the flow property for both K and K', we deduce that $K_{0,t} \stackrel{law}{=} K'_{0,t}$. $\square$

Remark 1. *Let φ be the coalescing flow constructed in Section 2. Then*

$$\varphi \stackrel{law}{=} \varphi^c. \tag{23}$$

As before this remains to show that $\varphi_{s,t} \stackrel{law}{=} \varphi^c_{s,t}$ conditionally to $\{\rho_s > t\}$. However the situation is easier here and we don't need the lemmas 6,7,8,9 and 10. For example

$$\eta^+_{s,t} = 1_{\{\varphi^c_{s,t}(1) \in \mathscr{C}^+\}} - 1_{\{\varphi^c_{s,t}(1) \in \mathscr{C}^-\}}$$

is independent of $\sigma(|\varphi^c_{s,u}(1)|, s \leq u \leq \rho_s)$ conditionally to $\{\rho_s > t\}$ where $|\cdot|$ is the distance to 1 since $\varphi^c_{s,\cdot}(1)$ is a Brownian motion on $\mathscr{C}$. Following Proposition 9, it is easy to check (23). In particular φ^c solves $(T_{\mathscr{C}})$.

4 Proof of Proposition 1

In this section, we will use the same notations as in Section 2. For $r \geq 0$, we denote $W^{\pm}_{0,r}$ simply by $W^{\pm}_r$. For all $a \in \mathbb{R}$ define

$$T_a = \inf\{r \geq 0 : W_r = a\}$$

and for all $b \geq 0$, define

$$\rho^{\pm}_b = \inf\{r \geq 0 : W^{\pm}_r = b\}.$$

We will further need the following

Lemma 11. *For all $a > 0, b > 0$ and $c < 0$, we have $\mathbb{P}(T_a < \rho^-_b \wedge T_c) > 0$.*

Proof. Fix $\eta \in]0, \frac{b}{2} \wedge (-c)[$ and let $k \geq 1$ such that $k\eta \geq a$. Now define the sequence of stopping times $(R_i)_{i \geq 0}$ such that $R_0 = 0$ and for $i \geq 0$,

$$R_{i+1} = \inf\{r \geq R_i : |W_r - W_{R_i}| = \eta\}.$$

Let $A = \cap_{i=1}^k \{W_{R_i} = W_{R_{i-1}} + \eta\}$. Then on A, $\sup_{r\le R_k} W_r = k\eta \ge a$ and for all $i \in [0, k-1], u \in [R_i, R_{i+1}]$,

$$W_u^- = \sup_{r\le u} W_r - W_u = \sup_{R_i \le s\le u} (W_s - W_u) \le 2\eta < b.$$

Moreover $\inf_{0\le r\le R_k} W_r > -\eta \ge c$. Since $A \subset \{T_a < \rho_b^- \wedge T_c\}$ and $\mathbb{P}(A) = \frac{1}{2^k}$, this proves the lemma. $\square$

Let $a > 0$. Since $\{T_a < \rho_a^- \wedge T_{-a}\} \subset \{T_a < \rho_a^-\}$, we deduce that $\mathbb{P}(T_a < \rho_a^-) > 0$. Obviously $\rho_a^+ \le T_a$. Since $W \overset{law}{=} -W$, we have $\mathbb{P}(\rho_a^+ < \rho_a^-) = \mathbb{P}(\rho_a^- < \rho_a^+) = \frac{1}{2}$. Remark also that

$$\rho_a^+ \wedge \rho_a^- = \inf\{r \ge 0 : W_r^+ + W_r^- = a\}.$$

This shows that on $\{\rho_a^+ < \rho_a^-\}$, we have $W_{\rho_a^+}^- = 0$ and similarly on $\{\rho_a^- < \rho_a^+\}$, we have $W_{\rho_a^-}^+ = 0$.

4.1 The case $l = \pi$

The case $l = \pi$ is the easier one.

Lemma 12. *With probability* 1*, for all* $z \in \mathscr{C}$*, we have*

$$\varphi_{0,\rho_\pi^+}(z) = -1, \quad K_{0,\rho_\pi^+}^{m^+,m^-}(z) = \delta_{-1}$$

and

$$\varphi_{0,\rho_\pi^-}(z) = 1, \quad K_{0,\rho_\pi^-}^{m^+,m^-}(z) = \delta_1.$$

Proof. The proof is obvious since $(\varphi_{0,\rho_\pi^+}(1), K_{0,\rho_\pi^+}^{m^+,m^-}(1)) = (-1, \delta_{-1})$ and $(\varphi_{0,\rho_\pi^-}(-1), K_{0,\rho_\pi^-}^{m^+,m^-}(-1)) = (1, \delta_1)$. $\square$

To prove Proposition 1, consider the sequences of stopping times given by $S_1 = \rho_\pi^+$ and for $k \ge 1$,

$$\begin{aligned} T_k &= \inf\{u \ge S_k : W_{S_k,u}^- = \pi\}, \\ S_{k+1} &= \inf\{u \ge T_k : W_{T_k,u}^+ = \pi\}. \end{aligned}$$

Then Lemma 12 implies that $(S_k)_{k\ge1}$ (resp. $(T_k)_{k\ge1}$) satisfies (1) (resp. (2)) of Proposition 1.

4.2 The case $l \neq \pi$

We fix $\delta > 0$ such that $0 < l - \delta < l + \delta < \pi$. For any $(\mathcal{F}^W_{0,\cdot})$-finite stopping time S and $a \in \mathbb{R}$ define

$$T_{S,a} = \inf\{r \geq S : W_{S,r} = a\}$$

and

$$\rho^-_{S,\delta} = \inf\{r \geq S : W^-_{S,r} = \delta\}.$$

Let

$$A_S = \{T_{S,2(\pi-l)} < \inf(T_{S,-l}, \rho^-_{S,\delta})\}.$$

Note that

$$A_S = \left\{\varphi_{S,\cdot}(e^{-il}) \text{ reaches } e^{il} \text{ before } 1 \text{ and before that } \varphi_{S,\cdot}(e^{il}) \text{ arrives in } e^{i(l+\delta)} \text{ or } e^{i(l-\delta)}\right\}.$$

Define the sequence $(\sigma_k)_{k\geq 0}$ of $(\mathcal{F}^W_{0,t})_{t\geq 0}$-stopping times by $\sigma_0 = 0$ and for $k \geq 0, \sigma_{k+1} = T_{\rho_{\sigma_k},2(\pi-l)}$ (note that $2(\pi - l) = \arg(e^{-il}) - \arg(e^{il})$). Then set, for $k \geq 0$,

$$C_k = \{W^+_{\sigma_k,\rho_{\sigma_k}} = l\} \cap A_{\rho_{\sigma_k}}.$$

Note that the events $\{W^+_{\sigma_k,\rho_{\sigma_k}} = l\}$ and $A_{\rho_{\sigma_k}}$ are independent. The following proposition describes what happens on C_k.

Proposition 10. *With probability* 1*, for all* $k \geq 0$*, on* C_k*, we have for all* $z \in \mathscr{C}$*,*

(i) $\arg(\varphi_{\sigma_k,\rho_{\sigma_k}}(z)) \in [l, 2\pi - l]$.

(ii) If $\arg(z) \in [l, 2\pi - l]$*, then* $\varphi_{\rho_{\sigma_k},\sigma_{k+1}}(z) = e^{il}$.

(iii) $\varphi_{\sigma_k,\sigma_{k+1}}(z) = e^{il}$.

(iv) $\varphi_{0,\sigma_{k+1}}(z) = e^{il}$ *and* $K^{m^+,m^-}_{0,\sigma_{k+1}}(z) = \delta_{e^{il}}$.

Proof. We take $k = 0$ (the proof is similar for all k). Denote ρ_0 simply by ρ and ρ^n_0 by ρ^n.
(i) Fix $z \in \mathscr{C}$. If $\tau_0(z) \leq \rho$, then $\varphi_{0,\rho}(z) \in \{\varphi_{0,\rho}(1), \varphi_{0,\rho}(e^{il})\}$. On C_0, we have $W^+_\rho = l$ and so $W^-_\rho = 0$ (see the lines after Lemma 11). Consequently $\varphi_{0,\rho}(e^{il}) = e^{il}$ and $\varphi_{0,\rho}(1) \in \{e^{il}, e^{-il}\}$.
Suppose $\rho < \tau_0(z)$, then necessarily $\arg(z) \in]l, 2\pi[$ and using that $W_\rho = l + \inf_{0\leq u\leq\rho} W_u$, we have

$$\varphi_{0,\rho}(z) = \exp(i(\arg(z) - W_\rho)) = \exp(i(\arg(z) - l - \inf_{0\leq u\leq\rho} W_u)).$$

Since $\rho < \tau_0(z)$, we have $\arg(z) - \inf_{0\leq u\leq\rho} W_u < 2\pi$ and therefore $\arg(\varphi_{0,\rho}(z)) < 2\pi - l$. It is also clear that $\arg(\varphi_{0,\rho}(z)) \geq l$ which proves the first statement.

(ii) Let $z \in \mathscr{C}$ with $\arg(z) \in [l, 2\pi - l]$. Then $\varphi_{\rho,\cdot}(e^{-il})$ arrives to e^{il} before 1 and this happens at time σ_1. Thus $\varphi_{\rho,\cdot}(z)$ reaches e^{il} before σ_1. Let n be the greatest integer such that $\rho_\rho^n (= \rho^{n+1}) \leq \sigma_1$. Then $\varphi_{\rho,\sigma_1}(z) = \varphi_{\rho^{n+1},\sigma_1}(Z)$ where $Z = \varphi_{\rho,\rho^{n+1}}(z)$. Clearly $\tau_{\rho^{n+1}}(Z) = \tau_\rho(z) \leq \sigma_1$. Therefore $\varphi_{\rho,\sigma_1}(z) = \varphi_{\rho^{n+1},\sigma_1}(e^{il})$. But $-W_{\rho,u} + 2(\pi - l) \geq W^-_{\rho,u}$ for all $u \in [\rho, \sigma_1]$ and so $W^-_{\rho,\sigma_1} = 0$. As $\rho^{n+1} \geq \rho$, we get $W^-_{\rho^{n+1},\sigma_1} = 0$. That is $\varphi_{\rho,\sigma_1}(z) = e^{il}$.

(iii) and (iv) are immediate from the flow property (Corollary 2) and (i), (ii). The result for K^{m^+,m^-} can be proved by following the same steps with minor modifications. $\square$

Since for all $k \geq 0$, σ_k is an $(\mathcal{F}^W_{0,t})_{t\geq 0}$-stopping time, the sequence $(C_k)_{k\geq 0}$ is independent. We also have $\mathbb{P}(C_k) = \mathbb{P}(C_0) = \mathbb{P}(A_0) \times \mathbb{P}(W^+_\rho = l)$ for all $k \geq 0$. By Lemma 11, $\sum_{k\geq 0} \mathbb{P}(C_k) = \infty$ and the Borel-Cantelli lemma yields $\mathbb{P}(\overline{\lim} C_k) = 1$. We deduce that with probability 1,

$$\varphi_{0,\sigma_k}(\mathscr{C}) = e^{il} \text{ and } K^{m^+,m^-}_{0,\sigma_k}(\mathscr{C}) = \delta_{e^{il}} \text{ for infintely many } k.$$

Lemma 13. *Let $(k_n)_{n\geq 0}$ be the sequence of random integers defined by $k_0(\omega) = 0$ and for $n \geq 0$,*

$$k_{n+1}(\omega) = \inf\{k > k_n(\omega) : \omega \in C_k\}.$$

Set $\sigma'_n = \sigma_{k_n}, n \geq 1$. Then $(\sigma'_n)_{n\geq 1}$ is a sequence of $(\mathcal{F}^W_{0,t})_{t\geq 0}$-stopping times such that a.s. $\lim_{n\to\infty} \sigma'_n = +\infty, \varphi_{0,\sigma'_n}(\mathscr{C}) = e^{il}$ and $K^{m^+,m^-}_{0,\sigma'_n}(\mathscr{C}) = \delta_{e^{il}}$ for all $n \geq 1$.

Proof. Remark that $C_k \in \mathcal{F}^W_{\sigma_{k+1}}$ for all $k \geq 0$. For all $n \geq 1$ and $t \geq 0$, we have

$$\{\sigma_{k_n} \leq t\} = \cup_{k\geq 1}\{\sigma_k \leq t, k_n = k\}.$$

It remains to prove that $\{k_n = k\} \in \mathcal{F}^W_{\sigma_{k+1}}$. We will prove this by induction on n. For $n = 1$, this is clear since $\{k_1 = 1\} = C_1$ and for $k \geq 2$,

$$\{k_1 = k\} = C_1^c \cap \cdots \cap C_{k-1}^c \cap C_k.$$

Suppose the result holds for n. Then for all $k \geq 2$,

$$\{k_{n+1} = k\} = \cup_{1\leq i\leq k-1} \left(\{k_n = i\} \cap C^c_{i+1} \cap \cdots C^c_{k-1} \cap C_k\right)$$

and the desired result holds for $n + 1$ using the induction hypothesis.

$\square$

We have proved Part (1) of Proposition 1 (for both φ and K^{m^+,m^-}). Part (2) can be deduced by analogy.

5 The support of K^{m^+,m^-} (Proof of Proposition 2)

In this section ρ_0^k and K^{m^+,m^-} will be denoted simply by ρ^k and K.

5.1 The case $l = \pi$

When m^+ and m^- are both different from $\frac{1}{2}(\delta_0 + \delta_1)$, a precise description of $\text{supp}(K_{0,t}(1))$ can be given as follows. Recall the definitions of the sequences $(S_k)_{k\geq 1}$ and $(T_k)_{k\geq 1}$ from Section 4.1 and set $T_0 = 0$. Then for all $k \in \mathbb{N}, t \in [T_k, S_{k+1}]$,

$$\text{supp}(K_{0,t}(1)) = \{e^{iW^+_{T_k,t}}, e^{-iW^+_{T_k,t}}\}$$

and for all $k \geq 1, t \in [S_k, T_k]$,

$$\text{supp}(K_{0,t}(1)) = \{e^{i(\pi+W^-_{S_k,t})}, e^{i(\pi-W^-_{S_k,t})}\}.$$

In fact, for all $s \leq t$,

$$\text{supp}(K_{s,t}(1)) = \{e^{iX_{s,t}}, e^{-iX_{s,t}}\},$$

with $X_{s,t}$ being the unique reflecting Brownian motion on $[0, \pi]$ (see [1]) solution of

$$X_{s,t} = W_{s,t} + L^0_{s,t} - L^\pi_{s,t}, \quad t \geq s,$$

and

$$L^x_{s,t} = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_s^t 1_{\{|X_{s,u}-x|\leq\varepsilon\}} du, \quad x = 0, \pi.$$

If $m^+ = m^- = \delta_{\frac{1}{2}}$, then K is a Wiener flow such that $K_{s,t}(1) = \frac{1}{2}(\delta_{e^{iX_{s,t}}} + \delta_{e^{-iX_{s,t}}})$ for all $s \leq t$.

5.2 The case $l \neq \pi$

From the definition of K, $K_{\rho^k,t}(z)$ is carried by at most two points for all $k \geq 0$, $t \in [\rho^k, \rho^{k+1}]$ and $z \in \mathscr{C}$. Using the flow property and the fact that $\lim_{k\to\infty} \rho^k = \infty$ a.s., it is therefore clear that a.s.

$$\forall t \geq 0, \ z \in \mathscr{C}, \ \text{Card supp } K_{0,t}(z) < \infty.$$

We assume in this section that m^+ and m^- are both distinct from $\frac{1}{2}(\delta_0 + \delta_1)$ (for the other case, see Remark 2 below).

Fix a decreasing positive sequence $(\alpha_k)_{k\geq 1}$ such that $\alpha_1 < \inf(l, 2(\pi - l))$. Now define $A_1 = \{\, W^+_{0,\rho^1} = l\}$ and for $k \geq 1$,

$$\begin{aligned} A_{2k} &= \{W^-_{\rho^{2k-1},\rho^{2k}} = l,\ \alpha_{2k} < \sup_{\rho^{2k-1}\leq u\leq \rho^{2k}} W_{\rho^{2k-1},u} < \alpha_{2k-1}\} \\ &= \{W^-_{\rho^{2k-1},\ \rho^{2k}} = l, -l + \alpha_{2k} < W_{\rho^{2k-1},\rho^{2k}} < -l + \alpha_{2k-1}\}, \\ A_{2k+1} &= \{W^+_{\rho^{2k},\rho^{2k+1}} = l,\ -\alpha_{2k} < \inf_{\rho^{2k}\leq u\leq \rho^{2k+1}} W_{\rho^{2k},u} < -\alpha_{2k+1}\} \\ &= \{W^+_{\rho^{2k},\rho^{2k+1}} = l,\ l - \alpha_{2k} < W_{\rho^{2k},\rho^{2k+1}} < l - \alpha_{2k+1}\}. \end{aligned}$$

We are going to prove the following

Proposition 11. *Let $C_0 = \Omega$ and $C_n = \cap_{i=1}^n A_i$ for all $n \geq 1$. Then for all $n \geq 0$,*

(i) $\mathbb{P}(C_n) > 0$,

(ii) Card supp $(K_{0,\rho^n}(1)) = n + 1$ *a.s. on* C_n.

Moreover a.s. for all $k \geq 0$,

(ii1) On C_{2k},

$$supp\left(K_{0,\rho^{2k}}(1)\right) = \{P_i^{2k}, 1 \leq i \leq 2k+1\},$$

with $\arg(P_i^{2k}) < \arg(P_{i+1}^{2k})$ *for all* $i \in [1, 2k]$,

$$P_1^{2k} = 1,\ P_2^{2k} = e^{2il} \ \text{and}\ \ P_{2k+1}^{2k} = e^{i(-l - W_{\rho^{2k-1},\rho^{2k}})}.$$

(Note that $\arg(P_{2k+1}^{2k}) < 2\pi - \alpha_{2k}$*.)*

(ii2) On C_{2k+1}, we have

$$supp\left(K_{0,\rho^{2k+1}}(1)\right) = \{P_i^{2k+1},\ 1 \leq i \leq 2k+2\},$$

with $\arg(P_i^{2k+1}) < \arg(P_{i+1}^{2k+1})$ *for all* $i \in [1, 2k+1]$,

$$P_1^{2k+1} = e^{il},\ P_2^{2k+1} = e^{i(2l - W_{\rho^{2k},\rho^{2k+1}})} \ \ \text{and}\ \ P_{2k+2}^{2k+1} = e^{-il}.$$

(Note that $\arg(P_2^{2k+1}) > l + \alpha_{2k+1}$*.)*

To prove this proposition, let first establish the following

Lemma 14. *Fix $0 < \alpha < \beta < l$ and define*

$$E = \{W_\rho^- = l, \alpha < \sup_{0\leq u\leq \rho} W_u < \beta\}$$

where $\rho = \inf\{r \geq 0 : \sup(W_r^+, W_r^-) = l\}$. *Then* $\mathbb{P}(E) > 0$.

Proof. Recall the definition of T_a from the begining of Section 4. Consider the event

$$F = \{T_\alpha < T_{\beta-l} < T_\beta\} \cap \{\text{after } T_{\beta-l}, W \text{ reaches } \alpha - l \text{ before } \beta - l + \alpha\}.$$

Using the Markov property at time $T_{\beta-l}$, we have $\mathbb{P}(F) > 0$. Note that ρ can be expressed as

$$\rho = \inf\{t \geq 0 : \sup_{0\leq u\leq t} W_u - \inf_{0\leq u\leq t} W_u = l\}.$$

On F, we have $T_{\beta-l} < \rho \leq T_{\alpha-l}$ and so $\alpha < \sup_{0\leq u\leq\rho} W_u < \beta$. Moreover, on F

$$W_\rho^+ = W_\rho - \inf_{0\leq u\leq t} W_u < \beta - l + \alpha - (\alpha - l) < l.$$

In other words $W_\rho^- = l$ which proves the inclusion $F \subset E$ and allows to deduce the lemma. □

Proof of Proposition 11 (i) The sequence $(A_i)_{i\geq 1}$ is independent and therefore we only need to check that $\mathbb{P}(A_n) > 0$ for all $n \geq 1$. But this is immediate from Lemma 14 for n even. By replacing W with $-W$, it is also immediate for n odd.

(ii) We denote the properties (ii1) and (ii2) respectively by $\mathcal{P}_{2k}$ and $\mathcal{P}_{2k+1}$. Let prove all the $(\mathcal{P}_i)_{i\geq 0}$ by induction. First $\mathcal{P}_0$ and $\mathcal{P}_1$ are clearly satisfied since $K_{0,0}(1) = \delta_1$ and supp $K_{0,\rho^1}(1) = \{e^{il}, e^{-il}\}$ on C_1. Suppose that all the $\mathcal{P}_i$ hold for all $0 \leq i \leq 2k-1$ where $k \geq 1$. On C_{2k}, $K_{\rho^{2k-1},t}(e^{-il}) \neq \delta_1$ for all $t \in [\rho^{2k-1}, \rho^{2k}]$ since for all $t \in]\rho^{2k-1}, \rho^{2k}]$, we have

$$-W_{\rho^{2k-1},t} < W^-_{\rho^{2k-1},t} \leq l.$$

Moreover, on C_{2k}, we have

$$\inf_{\rho^{2k-1}\leq t\leq\rho^{2k}} (2l - W_{\rho^{2k-2},\rho^{2k-1}} - W_{\rho^{2k-1},t}) = l - W_{\rho^{2k-2},\rho^{2k-1}} - W_{\rho^{2k-1},\rho^{2k}} > l.$$

Thus

$$K_{\rho^{2k-1},t}(P_2^{2k-1}) = e^{i(2l - W_{\rho^{2k-2},\rho^{2k-1}} - W_{\rho^{2k-1},t})} \neq e^{il}$$

for all $t \in [\rho^{2k-1}, \rho^{2k}]$ so that $\mathcal{P}_{2k}$ holds. Similarly, on C_{2k+1}, $K_{\rho^{2k},\cdot}(e^{2il})$ cannot reach $\delta_{e^{il}}$ before ρ^{2k+1} since for all $t \in]\rho^{2k}, \rho^{2k+1}]$,

$$W_{\rho^{2k},t} < W^+_{\rho^{2k},t} \leq l.$$

Moreover, on C_{2k+1},

$$\sup_{\rho^{2k}\leq u\leq\rho^{2k+1}} (2\pi - l - W_{\rho^{2k-1},\rho^{2k}} - W_{\rho^{2k},u}) = 2\pi - (W_{\rho^{2k-1},\rho^{2k}} + W_{\rho^{2k},\rho^{2k+1}}) < 2\pi.$$

Thus, on C_{2k+1}, $K_{\rho^{2k},t}(P^{2k}_{2k+1}) \neq \delta_1$ for all $t \in [\rho^{2k}, \rho^{2k+1}]$ and $\mathcal{P}_{2k+1}$ easily holds.

Remark 2. *When $m^+ \neq m^-, m^- = \frac{1}{2}(\delta_0 + \delta_1)$, by considering*

$$E_{2i-1} = A_{2i-1} \text{ and } E_{2i} = A_{2i} \cap \{K_{\rho^{2i-1},\rho^{2i}}(e^{il}) = \delta_1\} \text{ for } i \geq 1,$$

and then $F_n = \cap_{1\leq i\leq n} E_i$, we similarly show that supp$(K_{0,t}(1))$ may be sufficiently large with positive probability.

Acknowledgement. The first author is grateful to Yves Le Jan who suggested to him this problem as a part of his Ph.D.

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